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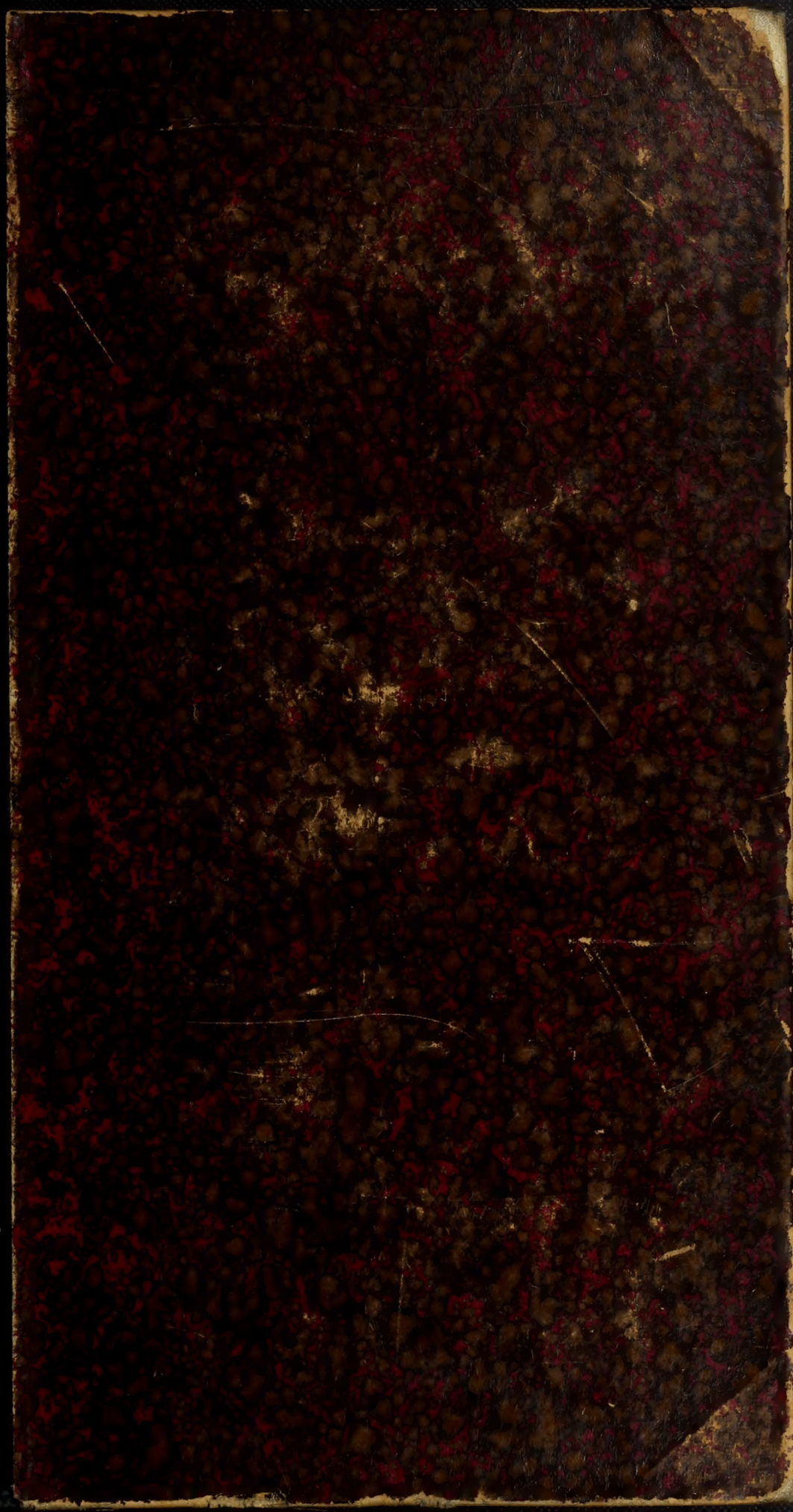

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# TERM BY TERM INTEGRATION OF INFINITE SERIES.

Lewis Omer

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in the
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THIS IS TO CERTIFY THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

Lewis Omer

ENTITLED Term by Term Integration of Infinite Series

IS APPROVED BY ME AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE DEGREE

OF Bachelor of Arts

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# Chapter I.

## Introduction

§ 1. General Statement of the Problem. — The problem with which this thesis is concerned is the integration of an infinite series term by term. Stated analytically it is this: — Let us have an infinite series

$$u_1 x + u_2 x + \cdots + u_n(x) + \cdots,$$

convergent towards a limit $f(x)$, for each value of $(x)$ lying within the finite interval $(a,b)$, that is

$$f(x) = u_1(x) + u_2(x) + \cdots + u_n x + \cdots; \; a \leqq x \leqq b,$$

and let us have $f(x)$ and each $u_n(x)$ continuous functions throughout the interval $(a,b)$. Let us have also

$$S_n(x) = u_1(x) + u_2(x) + \cdots u_n(x),$$

UIUC

$S_n(x)$ is then a continuous function in $(x)$. Then

$\int_a^b f(x)\,dx$ - is the same as

$\int_a^b \left(\lim_{n=\infty} S_n(x)\right) dx$. The term by term integration, namely

$\int_a^b u_1(x)\,dx + \int_a^b u_2(x)\,dx + \cdots,$

is the same as

$\lim_{n=\infty} \left[\int_a^b S_n(x)\,dx\right].$

Therefore, the problem may be stated as the question of whether or not

$$\int_a^b \left[\lim_{n=\infty} S_n(x)\right] dx = \lim_{n=\infty} \left(\int_a^b S_n(x).dx\right),$$

or, as to whether or not the integral of the sum of a series is equal to the sum of the integrals term by term.

We may study the problem from a geometrical standpoint, and it is the purpose here to use geometrical methods as much as possible. For the expression

$\int_a^b S_n(x)\,dx$ represents the area

between the curve, the axis of x, and the two ordinates for the values $x=a$, and $x=b$, where $S_n(x)$ is an approximation curve* of the series.

In a similar manner, the integral

$\int_a^b [\lim_{n=\infty} S_n(x)]\,dx$ represents the area

between the limiting curve $f(x)$, the x axis and the two ordinates taken as before. Geometrically, our problem is, under what conditions will the area between the approximation curve $S_n(x)$, the axis of $(x)$ and the assigned limiting values,

---

* See page 14, § 2 of this chapter.

approach as a limit, as $n$ increases indefinitely, the area bounded by the limiting curve $f(x)$, the $x$ axis, and the assigned ordinates.

Example. Given the infinite series

$$S_n^*(x) = \frac{x^2}{1+x^2} + \frac{4x^2}{1+4x^2} + \frac{x^2}{1+x^2} + \cdots\cdots$$

$$+ \frac{n^2x^2}{1+n^2x^2} - \frac{(n-1)^2x^2}{1+(n-1)^2x^2}; \quad 0 < x \leq 1,$$

where

$$f(x) = \lim_{n=\infty} S_n(x) = \lim_{n=\infty} \frac{n^2x^2}{1+n^2x^2},$$

The general term of our series may be written

$$u_n(x) = \frac{n^2x^2}{1+n^2x^2},$$

which is composed of two factors $(n^2x^2)$ and $\frac{1}{1+n^2x^2}$ each of which is continuous for every value of $x$ within the given interval and therefore their product is continuous. We

* See § 2, Chapter II of this thesis.

have

$$(1)\int_0^1 S_n(x)\,dx = \sum_1^n \int_0^1 u_n(x)\,dx = \int_0^1 \frac{x^2dx}{1+x^2} +$$

$$\int_0^1 \frac{4x^2dx}{1+4x^2} - \int_0^1 \frac{x^2dx}{1+x^2} + \cdots \doteq \int_0^1 \frac{(n-1)^2x^2dx}{1+(n-1)^2x^2}$$

$$\doteq \int_0^1 \frac{n^2x^2\cdot dx}{1+n^2x^2} = \int_0^1 \frac{x^2dx}{\frac{1}{n^2}+x^2} =$$

$$\left[(x) - \frac{1}{n^2}\int_0^1 \frac{dx}{\left(\frac{1}{n^2}+x^2\right)}\right]_0^1.$$

$$\text{But } \int_0^1 \frac{dx}{\left(\frac{1}{n^2}+x^2\right)} = \left[\frac{1}{\sqrt{\frac{1}{n^2}}}\tan^{-1}x\sqrt{\frac{1}{n^2}}\right]_0^1.$$

Then we know that equation (1) is equal to

$$\left[x - \frac{1}{n}\tan^{-1}\frac{x}{n}\right]_0^1 = \left[1 - \frac{1}{n}\tan^{-1}\frac{1}{n}\right],$$

which expression, as $n \doteq \infty$, reduces to 1. Therefore the limit approached by the sum of the areas under the various approximation curves, and within the assigned limits is 1.

Likewise

$$f(x) = \lim_{\substack{n \doteq \infty \\ x \neq 0}} \frac{n^2 x^2}{1 + n^2 x^2} = \lim_{\substack{n \doteq \infty \\ x \neq 0}} \frac{1^2}{\frac{1}{n^2 x^2} + 1} = 1.$$

Therefore.

$$\lim_{n \doteq \infty} \int_0^1 S_n(x)\, dx = \int_0^1 \lim_{n \doteq \infty} S_n(x)\, dx = \int_0^1 f(x)\, dx = 1.$$

This is shown geometrically in Fig. 1. The curves given there represent the various values of $S_n(x)$ as $n$ varies, and the values of $n$ taken are plotted upon the axis $\frac{1}{n}$. These approximation curves approach some curve as a limit as $n$ is indefinitely increased, and the curve approached is the curve representing $f(x)$. Then in this case, the integral of the limit is the same as the sum of the integrals term by term.

However, if the infinite series

$$S_n(x) = x\varepsilon^{-x^2} + [2x\varepsilon^{-2x^2} - x\varepsilon^{-x^2}] + \cdots$$

$$+ [n\, x\varepsilon^{-nx^2} - (n-1)x\varepsilon^{-(n-1)x^2}]; \quad 0 \leqq x \leqq 1,$$

be considered, where the $u(x)$'s are continuous functions of $(x)$ and where

$$f(x) = \lim_{n \doteq \infty} S_n(x),$$

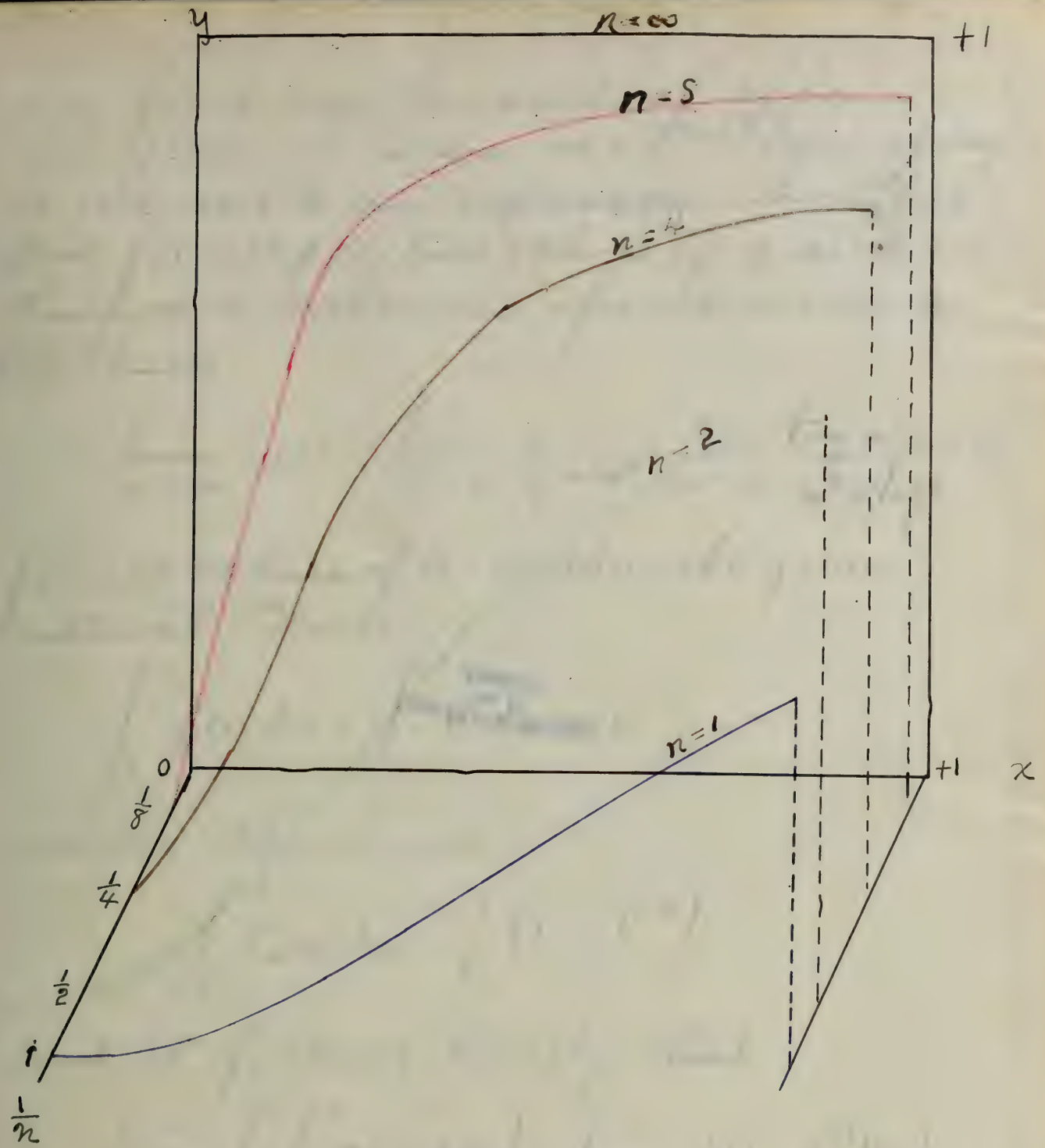



FIG. I

$$S_{(n)} = \frac{n^2x^2}{1+n^2x^2}$$

we find another state of affairs. The general term $n x \varepsilon^{n x^2}$ can always be reduced to an expression which is positive for all positive values of $x$ and is therefore a continuous function in $x$. We have

$$\lim_{n=\infty} S_n(x) = \lim_{n=\infty} \frac{n x}{\varepsilon^{n^2 x^2}} = \lim_{n=\infty} \frac{\log n x}{n^2 x^2 \log \varepsilon} = 0,$$

for all values of $(x)$ within the given interval. Hence

$$\int_0^1 f(x)\,dx = \int_0^1 0 \cdot dx = 0.$$

(On the other hand,

$$\int_0^1 S_n(x)\,dx = \frac{1}{2}(1 - \varepsilon^n)$$

and it follows directly that

$$\lim_{n=\infty}\left[\int_0^1 S_n(x)\,dx\right] = \lim_{n=\infty} \cdot \frac{1}{2}(1 - \varepsilon^n) = \frac{1}{2}.$$

Therefore

$$\int_0^1 f(x)\,dx = \int_0^1 \lim_{n=\infty} S_n(x)\,dx \neq \lim_{n=\infty} \int_0^1 S_n(x)\,dx =$$

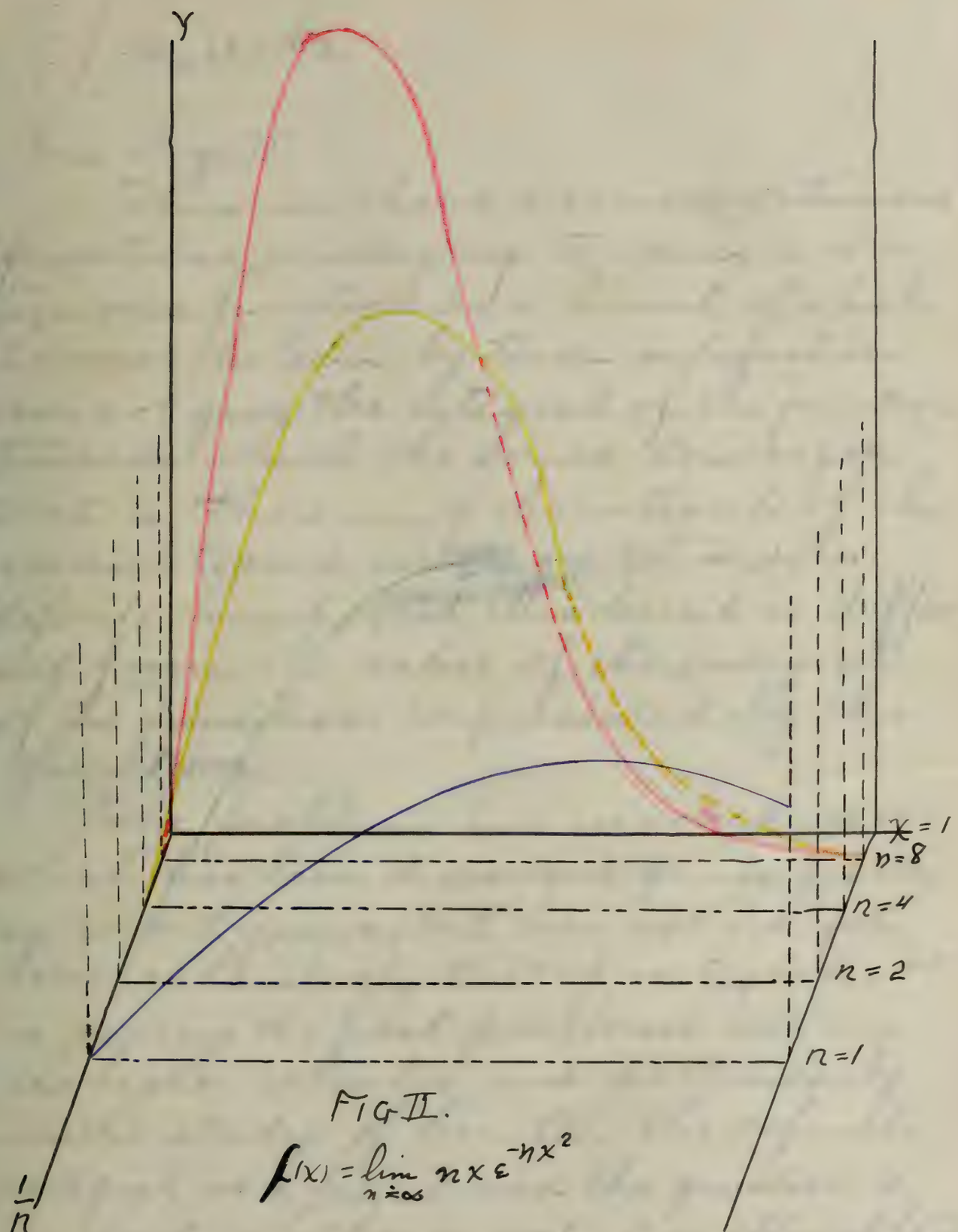
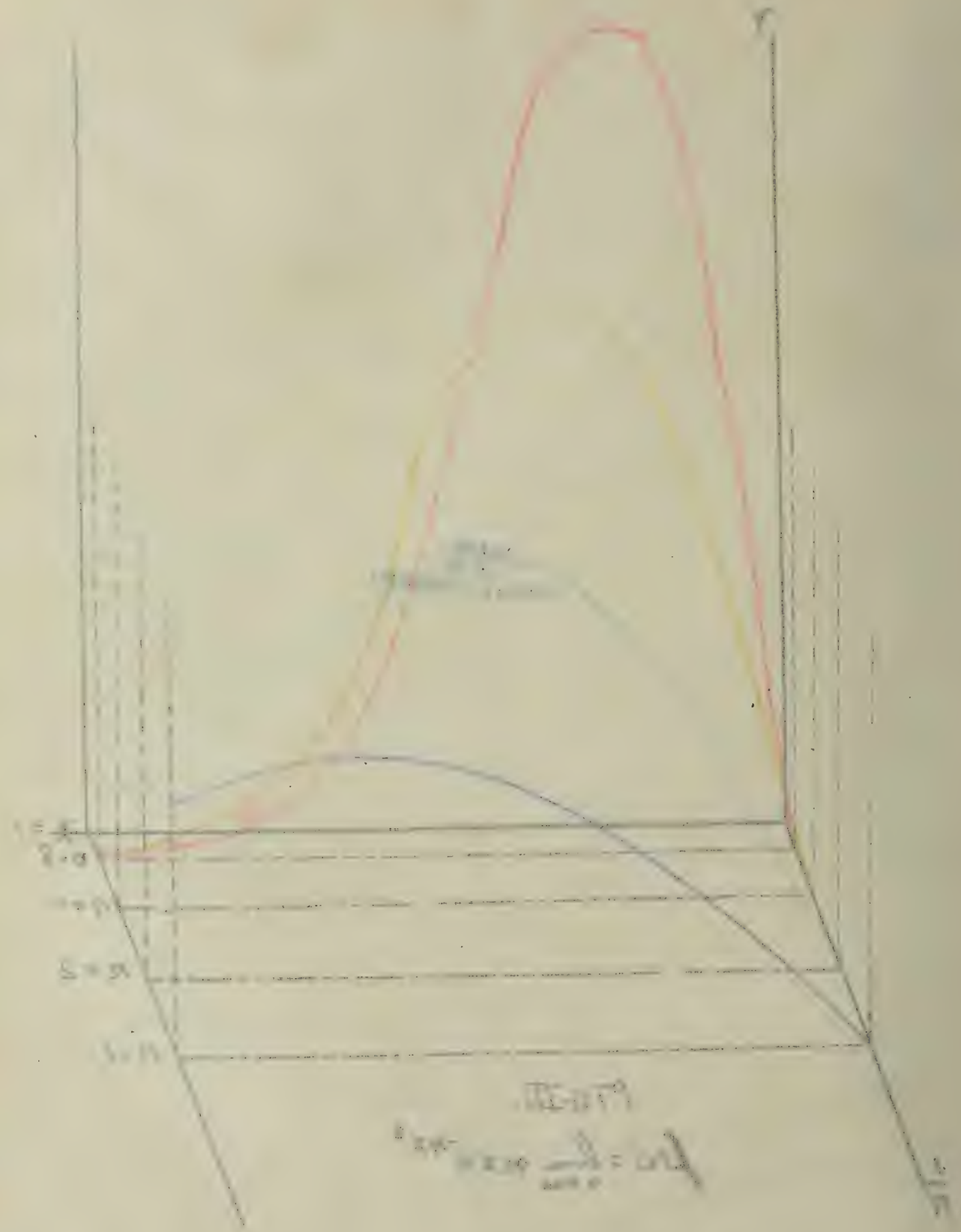



FIG II.

$$f(x) = \lim_{n \to \infty} n x \varepsilon^{-n x^2}$$



$$\sum_{1}^{\infty} \int_{0}^{1} u_n(x)\,dx,$$

See Fig II

Thus we have a series of continuous functions, converging toward a continuous function as a limit, of which however the term by term integration does not give the integral of the function towards which the series converges; that is, the sum of the integrals of the several terms converges towards a definite limit, but this limit is different from the value of the integral of the function represented by the function.

The problem here involved is one which has been discussed quite freely in recent years, but has not as yet been exhaustively treated in English. It is among the first problems met in the higher calculus, and particularly in the study of limits. The definite integral is a limit and the question is, in fact, whether or not it is allowable to reverse the order in certain twice-taken

limits. We find that many writers of elementary texts on calculus fail to give sufficient conditions for the integration of a series term by term, and a study of the manner in which this problem is treated in the commoner texts would be quite interesting.

Among the leading American writers upon this and kindred subjects is W. F. Osgood of Harvard, and his papers are from a geometrical standpoint among the best presentations of the question to be found. His papers deal more especially with this problem are those given before the American Mathematical Society at the summer meeting 1896, and published in the Bulletin, Vol. III, No. 2 pp. 59-86; and in the American Journal of Mathematics, Vol. XIX, 1897.

This subject was discussed previously by Prof. C. Arzela, an Italian mathematician, and it was because of some error in his development, that Prof. Osgood was led to treat the subject geometrically. Prof.

Osgood has discussed in the above papers the question under the following conditions; namely

(1) when $u(x)$ is continuous in $(x)$ within the given interval $(a,b)$,

(2) $f(x) = \underset{n=\infty}{L} s_n(x)$ for each value of $(x)$ in the interval $(a,b)$,

(3) when $f(x)$ is continuous and finite.

Arzela has since discussed the same problem again in "Sulla Serie di Funzione", but under much broader conditions which include Prof. Osgood's work as a special case; viz.

(1) where $f(x) = \underset{x=\infty}{L} s_n(x)$ for every $(x)$ in $(a,b)$,

(2) where $u_x(x)$ is integrable in $(a,b)$,

(3) where $f(x)$ is integrable in the given interval $(a,b)$.

§ 2 General Definitions. — (a) Definition of our series. We define our series as follows; Let

$$S_n(x) = u_1(x) + u_2(x) + \cdots + u_n x,$$

where $S_n(x)$ converges toward a limit $f(x)$ for each constant value of $(x)$ within the given interval, that is, where

$$fx = \lim_{n=\infty} S_n(x) \; ; \; a \leqq x \leqq b$$

Furthermore, let us assume that $f(x)$ and the $u$'s are each continuous functions of $(x)$ throughout the interval assigned. Then $S_n(x)$ not only converges for each value of $(x)$ within the given interval, but is for all finite values of $n$ a continuous function of $x$ within the same interval.

Geometrically, the infinite series defined as above represents the curve corresponding to the function $f(x)$ throughout the interval of convergency. The $S_n(x)$ represents a curve in the

same interval. Another statement of the question to be discussed is as to whether or not the curve representing $f(x)$ is the limit approached by the various approximation curves represented by $S_n(x)$ as $n$ shall increase indefinitely.

(b) Uniform convergence and non-uniform convergence. Let the general form of our infinite series be as previously defined

$$u_1(x) + u_2(x) + u_3(x) + \cdots + u_n(x) + \cdots,$$

and let

$$S_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x),$$

for all finite positive integral values of $n$.

We find that we have in fact a problem in double limits, and the infinite series may therefore be taken as a special case of a function of two variables. Many of the results and theorems relating to functions of two variables may then be directly

applied to the study of infinite series.

For example, when $f(x\,z)$ is a function of two variables defined within the region $a \leqq x \leqq b$, $z_0 \leqq z \leqq z_1$ we may define, as follows, the values of the function on the boundary of the region, namely, for $z = z_0$

$$f(x\,z_0) = \lim_{z \doteq z_0} f(x\,z).$$

(We have in an infinite series a parallel case,

$$f(x) = \lim_{n \doteq \infty} S_n(x),$$

where $S_n$ is defined for integral values of $n$. Instead of the variable $z$ we have $n$, and so have the two variables $x$ and $n$. This enables us to readily consider infinite series from a geometrical standpoint. We are able to study the infinite series as a function in three dimensional space exactly as is done in the study of two variables. In place of $z$ we have $n$, and consider the axis of $\frac{1}{n}$ instead of the $z$ axis. This gives us a method of seeing geometrically

the behavior of the approximation curves as $n$ approaches $\infty$, that is as $\frac{1}{n}$ approaches zero. By an approximation curve one means the curve which represents the function $S_n(x)$ when $n = 1, 2, 3, \ldots\ldots$ We have the various approximation curves as shown in any representation of a series in three dimensional space See FIG 2. The limit of the approximation curves may or may not be the curve representing the infinite series. When it is the same as the curve representing the infinite series, that is representing the limiting function, we have a necessary geometrical requirement for uniform convergence, that is to say, the approximation curves must go over into the curve of the limiting function. Let us define uniform convergence of an infinite series as follows;

<u>When for every value of $n$ greater than some definite $n$, we have simultaneously for all values of $x$ within the given interval the inequality</u>

$$|f(x) - S_n(x)| < \varepsilon, \quad a \leqq x \leqq b,$$

where $\varepsilon$ is an arbitrarily small positive number, then $S_n x$ is said to converge uniformly within the given interval $(a, b)$. It is necessary that we choose our values in the following order; first any $\varepsilon$ as small as we may choose which, when once chosen, remains fixed; then our $n$ such that the above equation holds, and lastly this inequality must hold simultaneously for all $(x)$'s within the given interval. This is in fact a geometrical definition and can be better understood when shown as represented in Fig. III. We have in the above figure the curve represented by $f(x)$, the limit approached by $S_n(x)$ as $n$ increases indefinitely and the curves represented by $S_{n'}(x)$ and $S_{n''}(x)$ respectively, where $n'$ and $n''$ are different values of $n$. The strip $2\varepsilon$ in width represents the space within which the projection of our curve $S_n(x)^*$ must fall. We have here again the condition that $\varepsilon$ must be chosen first, then for this $\varepsilon$ we must find an $n$ such that the projection of that curve

* For all values of $n > n_1$.

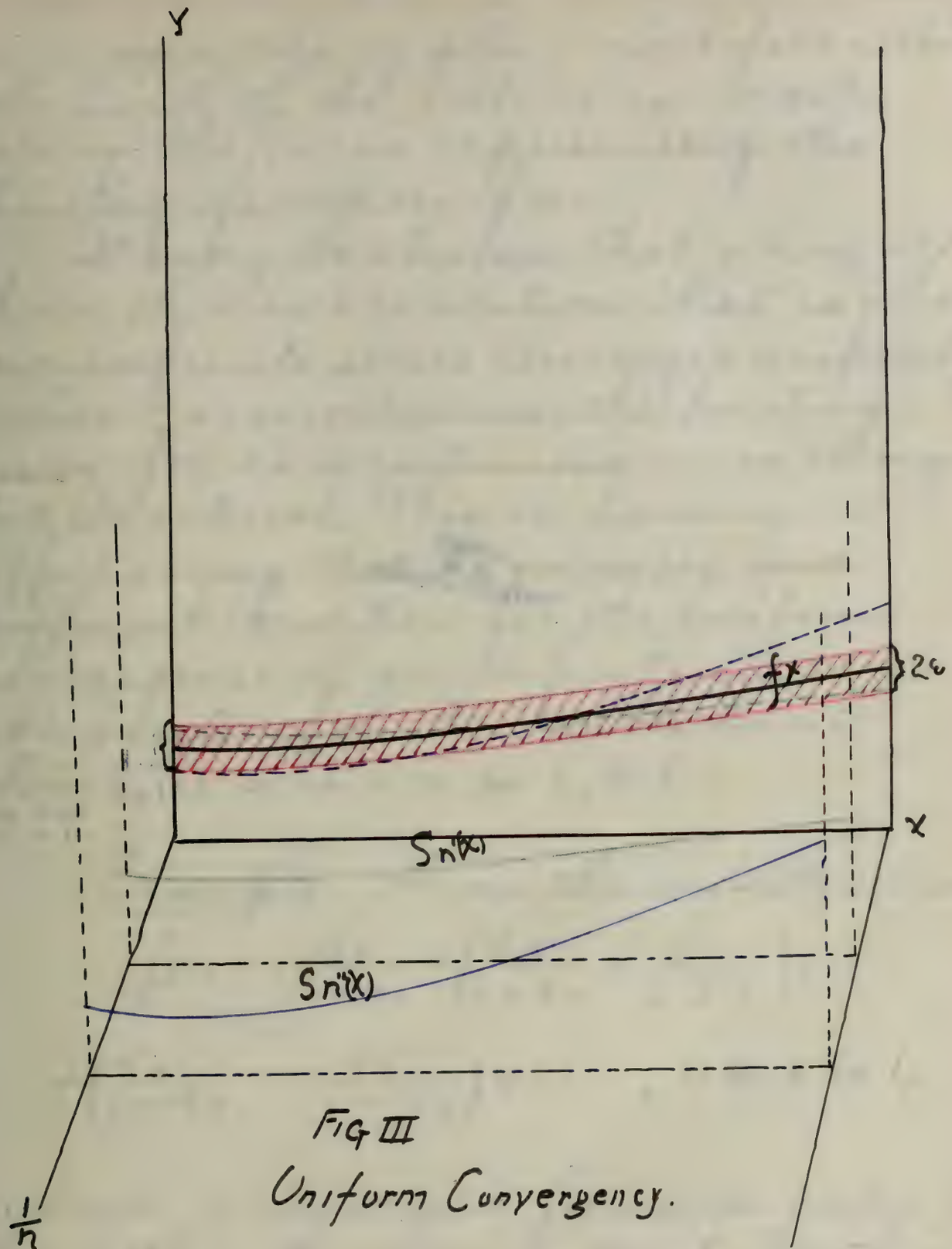



FIG III

Uniform Convergency.

Fig III

Uniform Compression

$S_n(x)$ upon the $xy$ plane will fall within the limits of the belt, $2\varepsilon$ in width, along the curve representing the limiting function $f(x)$.

It may be shown that when we have the above condition, that is when our infinite series converges uniformly when $S_n(x)$ is continuous, the limiting curve $f(x)$ is a continuous curve throughout the interval. This is however in effect saying that the necessary and sufficient condition for the uniform convergence of an infinite series is the existence of the double limit $\lim_{\substack{n=\infty \\ x=0}} S_n(x)$ where $a \leqq x_0 \leqq b$.

Example. Given the infinite series

$$f(x) = \frac{2x}{1+x} + \left(\frac{4x}{1+4x} - \frac{2x}{1+x}\right) + \cdots$$

$$+ \left(\frac{5x}{1+9x} - \frac{4x}{1+4x}\right) + \cdots ; \quad 0 \leqq x \leqq 1,$$

which is convergent for each value of $(x)$ within the given interval. In this series we have

$$S_n(x) = \frac{n+2}{n}\left(\frac{nx}{1+n^2x}\right).$$

Considering an infinite series as only a special case of a function of two variables, we test the uniform convergence of this series within the given interval by the method of double limits.* That is, we will see whether or not the double limit exists and equals the value of the limiting function for each value of (x), the end points included. Let the given interval be 10,11 and

$$\lim_{n \doteq \infty} S_n(x) = \lim_{\substack{n \doteq \infty \\ x \doteq 0}} \frac{n+2}{n}\left(\frac{nx}{1+n^2x}\right).$$

Putting $n = \frac{1}{x}$ then

$$\lim_{x \doteq 0} \frac{\frac{1}{x}+2}{\frac{1}{x}}\left(\frac{1}{1+\frac{1}{x}}\right) = \lim_{x \doteq 0} \frac{x+2}{x+1} = 2,$$

Putting $n = \frac{2}{x}$, however, we get

$$\lim_{x \doteq 0} \frac{\frac{2}{x}+2}{\frac{2}{x}}\left(\frac{2}{1+\frac{4}{x}}\right) = \lim_{x \doteq 0}(1+x)\cdot\frac{2}{1+\frac{4}{x}}$$

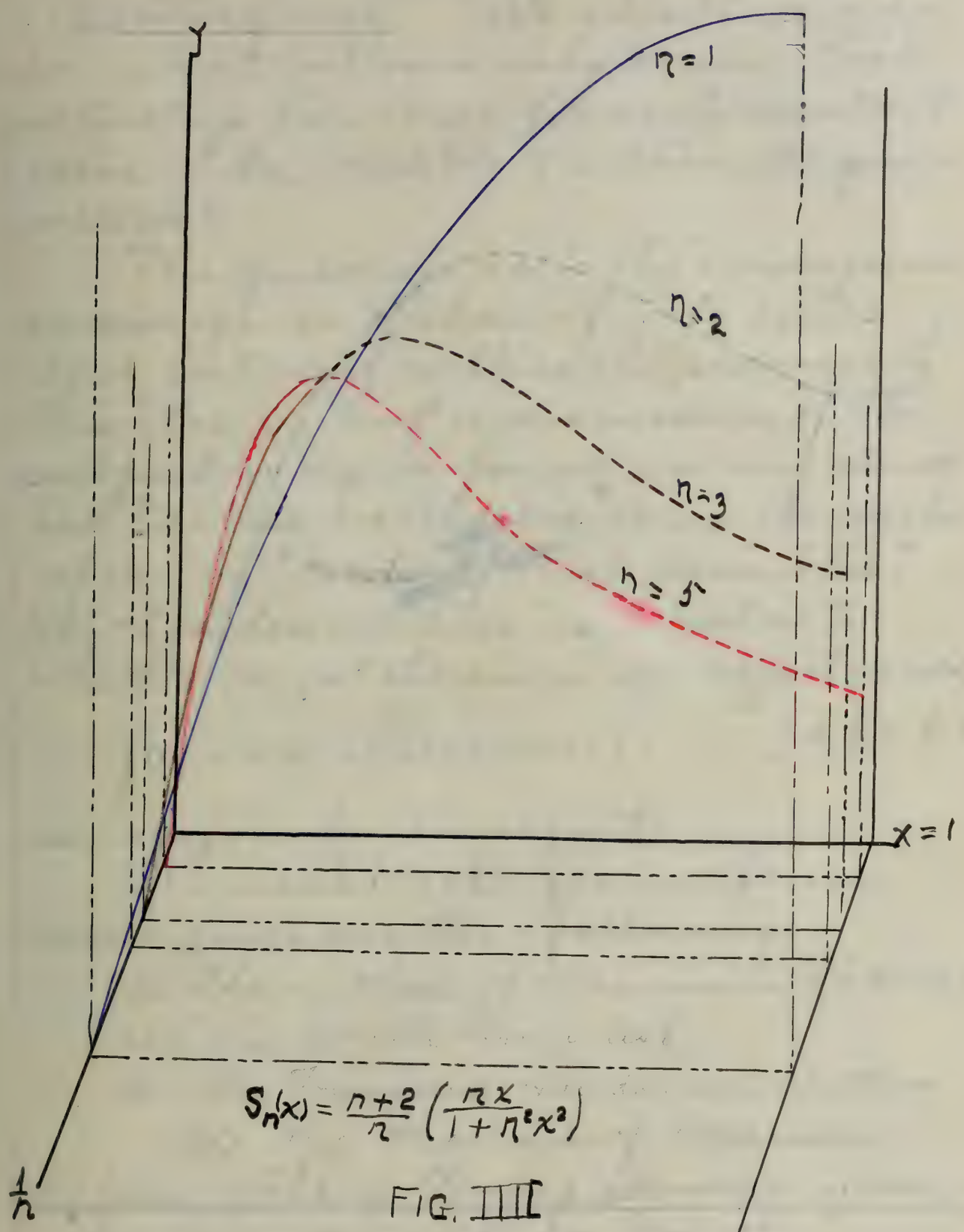
---

* See Dr. Townsend's Thesis, published at Göttingen 1900, Chapter I pp. 3, § 2. Theorem I

$$= \lim_{x \doteq 0} \left( \frac{2 + 2x}{1 + 4x} \right) = \frac{2 + 0}{1 + \frac{4}{x}} = \frac{2 + 0}{1 + \infty} = 0.$$

Thus by the two approaches to the origin we get different limiting values, and therefore at the origin the double limit $\lim_{\substack{x \doteq 0 \\ n \doteq \infty}}$ does not exist and our series does not converge uniformly within the given interval.

That the given infinite series converges non-uniformly is seen also from the manner in which the approximation curves approach and finally coincide in the limit with the curve representing the limiting function. We readily see that they do not fall within the geometrical requirements of uniform convergency given on page 14, FIG. III. The function represented by the series is continuous but the series itself is non uniformly convergent. For a uniformly convergent series see Example 1. Chapter II, § 1.




$$S_n(x) = \frac{n+2}{n}\left(\frac{\pi x}{1+\pi^2 x^2}\right)$$

FIG. IIII

FIG III

## § 3. Tests of Uniform Convergence.

We shall consider in what follows only series which actually converge for each constant value of the variable $x$ within the given interval.

The question as to the convergence or divergence of the infinite series does not come within the province of this thesis, but a discussion of the subject may be found in any standard treatise on Algebra,* or in the works upon the "Functions of Real Variables." What is desired here is a test as to whether or not the given convergent series

$$f(x) = u_1(x) + u_2(x) + u_3(x) + \cdots\cdots ; \quad a \leqq x \leqq b$$

is uniformly convergent.

The usual tests for uniform convergence are the following;

(a) The method of Maximum Values,

(b) The Double Limit test,

(c) The Townsend-Bendixson method.

(a) The Theorem of Maximum Values; The following theorem gives a

*See Chrystal's Algebra. Vol. II.

sufficient condition for a uniformly-convergent infinite series.

"If a set of constant positive quantities, $M_1 M_2 M_3 \cdots$ can be found such that

(1) $|u_i(x)| \leqq M_i$ ; $a \leqq x \leqq b$,

and

(2) the series $\sum_{i=1}^{\infty} M_i$ converges

then the $u(x)$ series will converge uniformly."*

Example: Given the infinite series.

$$f(x) = \frac{4}{\pi}\left(\frac{\sin x}{(1)^2} - \frac{\sin^3 x}{3^2} + \frac{\sin 5x}{5^2} + \cdots + \frac{\sin nx}{n^2} + \cdots\right); \quad 0 \leqq x \leqq 1.$$

We have maximum values of the terms of our series when the $\sin nx, \cdots$ is 1. Applying the above test for uniform convergence, we have

$$\sum_1^{\infty} |M_k| = \frac{4}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) < \frac{4}{\pi}\sum_1^{\infty}\frac{1}{n^2}.$$

This last is a well known convergent series, hence the series which we

* See Dini, Theorie der Functionen einer Veränderlichen, § 140.

are testing is uniformly convergent in the interval $(0, \frac{\pi}{2})$.

(b) The Double Limit Test. The following theorem is the one upon which the double limit test of uniform convergence depends.

"When the given series $S_n(x)$ converges for each value of $x$ within the given interval, then

The necessary and sufficient condition that a series converges uniformly in a given interval $(a,b)$ is that the double limit $\lim_{\substack{x=x_0 \\ n=\infty}} S_n(x)$ exists and equals $f(x_0)$ for each value of $x_0$ within the given interval, end points included."

The infinite series

$$f(x) = u_1(x) + u_2(x) + u_3(x) + \cdots; \quad a \leqq x \leqq b,$$

as defined above, is a special case of a function of two variables, and the theorems of double limits* may be applied without violation of the laws

*For proof of the corresponding theorems in Double Limits, see Notes of Lectures on Function Theory (Real Variable) Chapter III §4, at the University of Illinois of the course given 1st semester 1901-2.

which govern infinite series. The proof of the above theorem is given in §3, Chapter V, of the Notes of Prof. Townsend's Lectures on Function Theory (Real Variable) 1901-2. See also Mr. W. Pitt Miller's thesis 1901.

From the theorem, it follows immediately that the tests for the existence of the double limit for all points of the interval are the tests likewise for uniform convergence. This gives us therefore a valuable method of testing for uniform convergence of a series.

Example. Given the infinite convergent series

$$f(x) = \frac{x^2-1}{x^2+1} + \left(\frac{x^2-4}{x^2+4} - \frac{x^2-1}{x^2+1}\right) + \cdots\cdots$$

$$; 0 \leqq x \leqq 1,$$

when the sum of the first $n$ terms of the series gives

$$S_n(x) = \frac{x^2 - n^2}{x^2 + n^2}.$$

We proceed to test for different

approaches to the origin. Let

$$x = 2n.$$

Substituting value of $x$ in the equation gives

$$\lim_{x=2n} \left( \frac{x^2 - n^2}{x^2 + n^2} \right) = \frac{4n^2 - n^2}{4n^2 + n^2} = \frac{3}{5},$$

Putting

$$x = 3n.$$

we have

$$\lim_{x=3n} \left( \frac{x^2 - n^2}{x^2 + n^2} \right) = \frac{9n^2 - n^2}{9n^2 + n^2} = \frac{4}{5},$$

that is, we have two approaches to the origin, $(x = 2n)$ and $(x = 3n$, and taking the limit along these curves gives different values of the limit. Therefore, the double limit does not exist and the series is not uniformly convergent within the interval (0,1) since double limit must exist at every point, the end points included.

(c) The Method of Derviatives, or the Townsend-Bendixon Method. — The

third method of testing for uniform convergence of an infinite series is that included in the following Theorem, developed by Prof. Townsend in his lectures upon "Theory of Functions of the Real Variable."

"If $S_n(x)$ for all values of $(x)$ within the interval $(a,b)$ and for all finite values of $n$ is capable of differentiation, and if

$$|S_n'(x)| < G,$$

where $G$ is some definite finite number, and furthermore, if $S_n(x)$ converges, that is if

$$f(x) = \lim_{n=\infty} S_n(x),$$

for a set of values of $(x)$ everywhere dense in $(a,b)$, then $S_n(x)$ converges uniformly in the interval $(a,b)$."

Example. Let us take as an example the infinite convergent series

$$f(x) = \frac{x}{x^2+1} + \left(\frac{2x}{4x^2+1} - \frac{x}{x^2+1}\right) + \left(\frac{3x}{9x^2+1} - \right) + \cdots; \quad 0 \leqq x \leqq 1.$$

Then for any $n$,

$$S_n(x) = \frac{nx}{n^2x^2+1}.$$

Proceeding to test the series by this method we see that

$$S_n'(x) = \frac{n(1-n^2x^2)}{(n^2x^2+1)^2}.$$

We must have,

$$\left| \frac{n(1-n^2x^2)}{(n^2x^2+1)^2} \right| < G,$$

But $S_n'(0) = n$

and therefore, for at least one value of the interval, namely $x=0$, we have

$$|S_n'(0)| > G,$$

When we take $n$ sufficiently large, whatever may have been the finite value previously given to $G$. We have then no finite $G$ existing and our series is non-uniformly convergent.

# Chapter II

## Integration of a Series Term by Term.

§1 *Integration of Uniformly Convergent Series Term by Term.* The general problem with which we are dealing, as stated before, concerns the integration of an infinite convergent series term by term. In this section we study the problem with reference to a special phase; that is, where we have a series which answers all of our general conditions, in which the sum of the first $n$ terms is continuous throughout the interval $(a,b)$ and converges towards a continuous limit as $n = \infty$, and where in addition we impose upon our series the extra condition of uniform convergence. In other words, we say that our series shall converge uniformly throughout the given inter-

val. Stated analytically our new conditions are these; Let us have given a series

$u_1(x) + u_2(x) + u_3(x) + \cdots\cdots + u_n(x) + \cdots\cdots,$

which converges uniformly towards a limit $f(x)$, where the variable $(x)$ lies between the limits $(a)$ and $(b)$, that is

$$f(x) = u_1(x) + u_2(x) + \cdots\cdots + u_n(x) + \cdots\cdots; \; a \leqq x \leqq b,$$

Let the $u$'s be continuous in $x$. Hence $f(x)$ is, because of the uniform convergence of the series, a continuous function throughout the interval $(a, b)$. We also let

$$S_n(x) = u_1(x) + u_2(x) + \cdots\cdots + u_n(x),$$

and have $S_n(x)$ a continuous function in $(x)$. Then, as before,

$$\int_a^b f(x)\,dx \text{ is the same as } \int_a^b \lim_{n=\infty} S_n(x)\,dx.$$

The term by term integration

$$\int_a^b u_1(x)\,dx + \int_a^b u_2(x)\,dx + \cdots\cdots,$$

is the same as

$$\lim_{n=\infty}\left[\int_a^b S_n(x)\,dx\right].$$

Then the question under consideration is that of whether or not

$$\int_a^b \left(\lim_{n=\infty} S_n(x)\right) dx = \lim_{n=\infty}\left[\int_a^b S_n(x)\,dx\right],$$

or as to whether or not the integral of the limit of a uniformly convergent series is the same as the limit of the sum of the integrals term by term.

Here we shall depend more than ever upon the geometrical methods of dealing with the problem, and perhaps a review of the geometrical interpretation of uniform convergence given previously would not be out of place.

Let us first consider the approximation curves of $S_n(x)$, as $n$ varies. These converge towards the limiting value of the series, as $n$ approaches infinity, which is represented by the curve $f(x)$.

Then if the inequality

$$|f(x) - S_n(x)| < \varepsilon;\ a < x \leq b,$$

be true, where $\varepsilon$ is some arbitrarily small positive value which when once chosen remains fixed, and where $n$ is any finite positive integral value greater than some definite finite number $n_1$, The series is uniformly convergent within the interval. In other words, any approximation curve from some definite value $S_{n_1}(x)$ on would if projected on the $x\,y$-plane fall within the belt $2\varepsilon$ within which the limiting curve $f(x)$ lies. See Fig III page 14 for discussion in full.

This enables us to see at once that the condition of uniform convergence of our series requires that the limit approached by the areas between the approximation curves and the axis of $(x)$ within the assigned interval $(a,b)$ must be the area under the limiting curve within the same limits This is stated in the following theorem.

The sufficient, but not necessary, condition that the integral of the limit

of an infinite series, convergent within a given interval, be equal to the limit of the sum of the term by term integrals of the series as n increased indefinitely, is that the series shall converge uniformly throughout the given interval of convergence.

Example. Let us consider the infinite series

$$f(x) = x^2 + \left(\frac{x^2}{2} - \frac{x^2}{1}\right) + \left(\frac{x^2}{3} + \frac{x^2}{2}\right) + \cdots; \quad 0 \leq x \leq 1.$$

*The sum of the first n terms of the series is

$$S_n(x) = \frac{x^2}{n}.$$

The problem is the question of whether or not

$$\int_0^1 \left(\frac{x^2}{n}\right) dx = \lim_{n=\infty} \left( \int_0^1 x^2 dx + \int_0^1 \frac{x^2}{2} dx \right.$$

$$\left. + \int_0^1 x^2 dx + \cdots \right); \quad 0 \leq x \leq 1.$$

We now proceed to test our series

* The series used here and the kind

$$S_n(x) = \frac{x^2}{n}$$

for the existence of the required conditions. Testing first for continuity we see that the series must be continuous for any finite positive integral value of $n$, for it may be

generally used throughout the article are known as telescope series, because of the fact that the terms cancel and our series can close up on itself like a telescope. One might object to them as being artificial, but, in effect it is only a question of the form in which the terms are written and we can express any series whatsoever in this form. For example the series

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots\cdots,$$

when written in the form

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots\cdots,$$

becomes a telescope series in which

$$S_n = \left(1 - \frac{1}{n+1}\right).$$

resolved into a product of $x^2$ times a constant. We then test for uniform convergence. At first glance one would test the series by the double limit method for the existence of the double limit, which is the necessary and sufficient condition for the uniform convergence of a series defined as the above. Our series may be tried for the existence of the double limit thus; putting $n = x$, we get

$$\lim_{\substack{x \doteq 0 \\ n = x}} \frac{x^2}{n} = \lim_{x \doteq 0} \frac{x^2}{x} = 0.$$

Again, we substitute $n = \frac{1}{x^2}$ and

$$\lim_{\substack{x \doteq 0 \\ n = \frac{1}{x^2}}} \frac{x^2}{n} = \lim_{x \doteq 0} x^4 = 0.$$

In this case we should have to test all different approaches and show that we have the same value for all methods of approach at the origin. This is impossible and in a case, when as here, the approximation curves approach the curve

represented by $f(x)$, we know geometrically that the double limit does exist We shall however apply another method. See § 4, Chapter I.

We use this time the Townsend-Bendixson method of derivatives. Our series is

$$S_n(x) = x^2 + \left(\frac{x^2}{2} - \frac{x^2}{1}\right) + \cdots + \left(\frac{x^2}{n} - \frac{x^2}{n-1}\right); 0 \leq x \leq 1.$$

Differentiating our series, we get the value

$$S_n'(x) = 2x + (x - 2x) + \cdots \cdots \left(\frac{2x}{n} - \frac{2x}{n-1}\right); 0 \leq x \leq 1.$$

The terms of the derivative converge uniformly for the inequality

$$|S_n'(x)| < 3$$

where $x$ is such as to make $S_n'(x)$ a maximum. Thus by the theorem under (c), § 3, Chapter I, the original series is uniformly convergent.

Looking at the series from a geometrical standpoint, we see that the approximation curves of the series approach the axis of $x$ as a limit; that

is to say, the areas under the various approximation curves of the series in Fig VI converge uniformly throughout the defined interval; and, there are no peaks and the curves fulfill the geometrical condition

$$|f(x) - S_n(x)| < \epsilon \; ; 0 \leq x \leq 1, \text{ all } n \geq m.$$

The problem is the question of whether or not the integral of the limit of the uniformly convergent series

$$S_n(x) = \frac{x^2}{n} \; ; \; 0 \leq x \leq 1,$$

is equal to the limit of the sum of the term by term integrals as $n$ increases indefinitely; as to whether or not the area under the curve representing $f(x)$ is the limit approached by the areas under the various approximation curves representing $S_n(x)$ as $n$ increases indefinitely.

However

$$\int_0^1 \lim_{n=\infty} \frac{x^2}{n} dx = \int_0^1 (0)\, dx = 0,$$

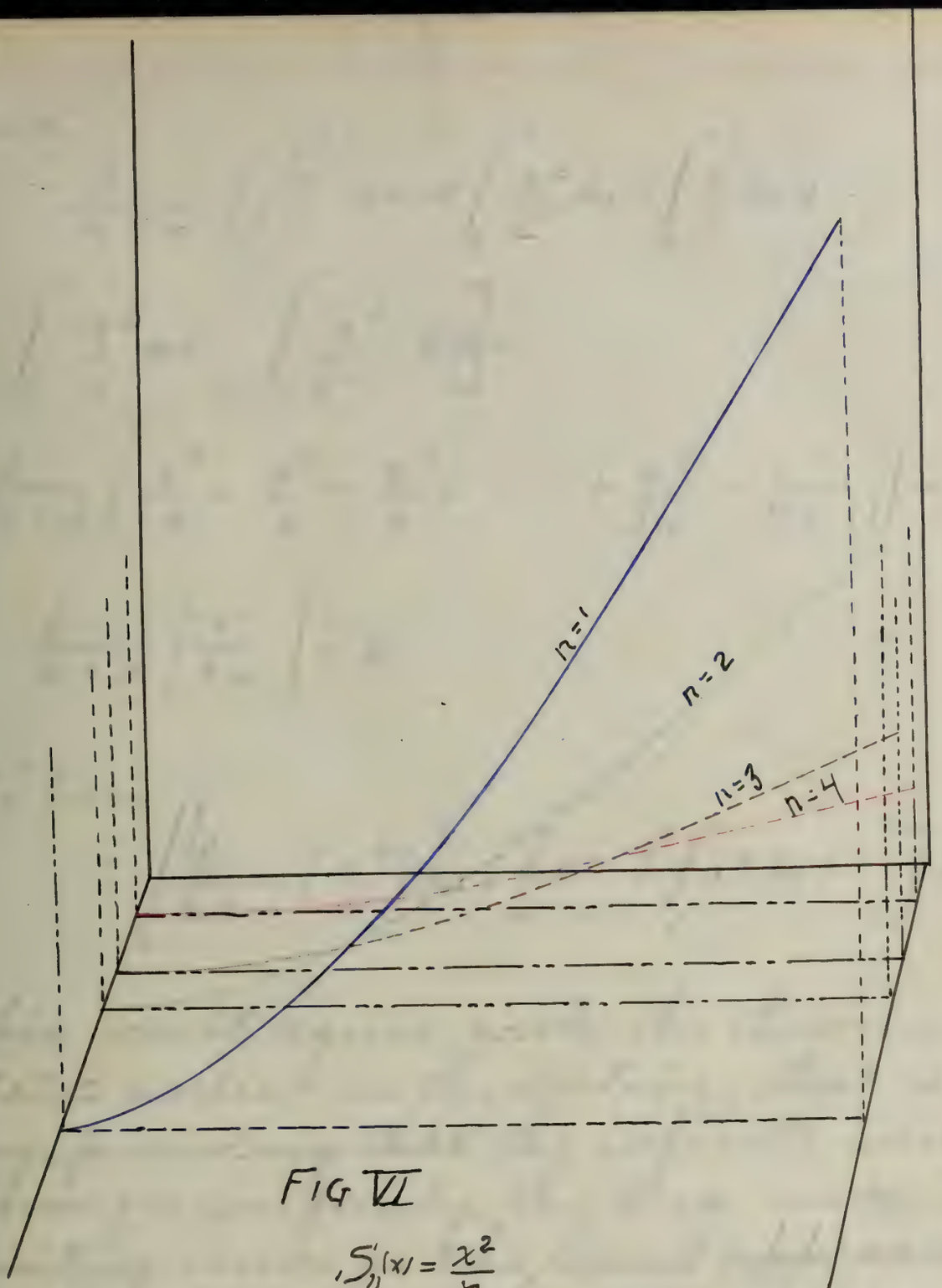



FIG VII

$$S_n(x) = \frac{x^2}{n}$$

and

$$\lim_{n=\infty}\left[\int_0^1 \frac{x}{1}^2 dx + \int_0^1 \frac{x^2}{2}\cdot dx - \int_0^1 \frac{x^2}{1} dx + \cdots\cdots\right.$$

$$\left. + \int_0^1 \frac{x^2}{n} dx - \int_0^1 \frac{x^2}{n-1}\cdot dx\right] =$$

$$\lim_{n=\infty}\left[\frac{x^3}{3} + \frac{x^3}{6} - \frac{x^3}{3} + \cdots\cdots + \frac{x^3}{3n} - \frac{x^3}{3(n-1)}\right] =$$

$$\lim_{n=\infty}\left[\frac{x^3}{3n}\right] = 0.$$

Then

$$\int_0^1 \lim_{n=\infty}\left(\frac{x^2}{n}\right) dx = \lim_{n=\infty}\left[\int_0^1 x^2 dx + \cdots\right] = 0.$$

This result agrees with the theorem stated earlier in the section; that is to say, providing that the infinite series converges uniformly, the area under the limiting curve is the limit approached by the areas under the various approximation curves as n increases indefinitely

Let us consider as another example the series,

$$f(x) = \frac{1}{2}\log(x^2+2) - \frac{1}{2.2}\log(2x^2+2)$$

$$+\frac{1}{2.2}\log(2x^2+2) - \frac{1}{2.3}\log(3x^2+2)+\cdots$$

$$+\frac{1}{2n}\log(nx^2+2) - \frac{1}{2(n+1)}\cdot\log(n+1x^2+2+\cdots;$$

$$0 \leqq x \leqq 1.$$

This is a series where

$$u_n(x) = \frac{1}{2n}\cdot\log(nx^2+2) - \frac{1}{2(n+1)}\cdot\log(n+1x^2+2).$$

This series being a telescopic series in which the first and last terms do not vanish, the sum of the first n terms of the series is

$$S_n(x) = \frac{1}{2}\log(x^2+2) - \frac{1}{2n+1}\cdot\log(n+1.x^2+2).$$

Studying the series by plotting the various approximation curves, we see, first that the series comes under the required condition, that is, is uniformly convergent having the $u(x)$ functions continuous; second, that our approx

imation curves converge upward toward a limit as n approaches infinity. The question again is whether or not the area under the limiting curve f(x) is the limit approached by the areas under the several approximation curves as n increases indefinitely. That the above condition does exist is seen by inspection of our series.

$$f(x) = \frac{1}{2}\log(x^2+2) - \frac{1}{2.2}\log(2x^2+2) + \cdots\cdots =$$

$$= \lim_{n \doteq \infty}\left\{\frac{1}{2}\log(x^2+2) - \frac{1}{2(n+1)}\log\overline{n+1}\,x^2+2\right\}.$$

Let us find this limit

$$\lim_{n \doteq \infty} S_n(x).$$

We know that

$$\lim_{n \doteq \infty} S_n(x) = \frac{1}{2}\log(x^2+2) - \lim_{n \doteq \infty}\frac{1}{2n+1}\log\sqrt{n+1}\,x^2+2.$$

By problem 13, page 26 of Pascal's Esercizi Di Calcolo Infinitesimale" we see that

$$\lim_{n \doteq \infty}\frac{\log(1+nx)}{n} = 0.$$

Then

$$\lim_{n=\infty} \frac{1}{n} \log(nx+2) = 0,$$

and

$$\lim_{n=\infty} S_n(x) = \frac{1}{2} \log(x^2+2), \quad 0 \leqq x \leqq 1.$$

But

$$S_n(x) = \frac{1}{2} \log(x^2+2) - \frac{1}{2n+1} \log(\overline{n+1}\, x^2+2)$$

and the problem is therefore the question of whether or not

$$\int_0^1 f(x)\,dx = \lim_{n=\infty} \int_0^1 S_n(x)\,dx,$$

or, this relation may be written as follows:

$$\int_0^1 \frac{1}{2}\log(x^2+2)\,dx = \int_0^1 \frac{1}{2}\log(x^2+2)\,dx + \lim_{n=\infty} \int_0^1 \frac{\log \overline{n+1}\,x^2+2)}{2(n+1)}\,dx.$$

This reduces to the question of whether or not the

$$\lim_{n=\infty} \int_0^1 \frac{1}{2(n+1)} \cdot \log(\overline{n+1}\, x^2+2)\,dx = 0.$$

We integrate as follows:

$$\frac{1}{2n+1}\int_0^{'} \log(\overline{n+1}\,x^2+2)\,dx.$$

Integrating by parts, we let

$$u = \log(\overline{n+1}\,x^2+2),$$

$$dv = dx \quad , \quad v = x$$

while

$$du = \frac{2\overline{n+1}\,x}{\overline{n+1}.x^2+2}.$$

Our integration gives the reduced expression

$$(1)\quad \frac{1}{2\ n+1}\ x.\log(\overline{n+1}\,x^2+2) - \frac{1}{2(n+1)}\int_0^{'} \frac{2\overline{n+1}\,x^2dx}{\overline{n+1}\,x^2+2}.$$

But the expression

$$-\int_0^{'} \frac{x^2dx}{\overline{n+1}x^2+2} = -\int_0^{'} \frac{dx}{n+1} + \int_0^{'} \frac{2.dx}{\overline{n+1}x^2+2}$$

$$= -\frac{x}{n+1} + \frac{4}{\sqrt{2}\,n+1}\ \tan^{-1}x\sqrt{\frac{2(n+1)}{2}}$$

Thus, our expression (1) reduces to the form

$$\left[\frac{1}{2n+1}.x\log(\overline{n+1}x^2+2 - \frac{x}{n+1} + \frac{4}{\sqrt{2}\,n+1}.\right.$$

$\tan^{-1} x\sqrt{\frac{2n+1}{2}}\Big]$

and the limit.

$$\lim_{n=\infty}\left\{\frac{1}{2n+1}.x\log(n+1\,x^2+2)-\frac{x}{n+1}+\frac{4}{\sqrt{2n+1}}.\tan^{-1}x\sqrt{n+1}\right\}=0.$$

Hence

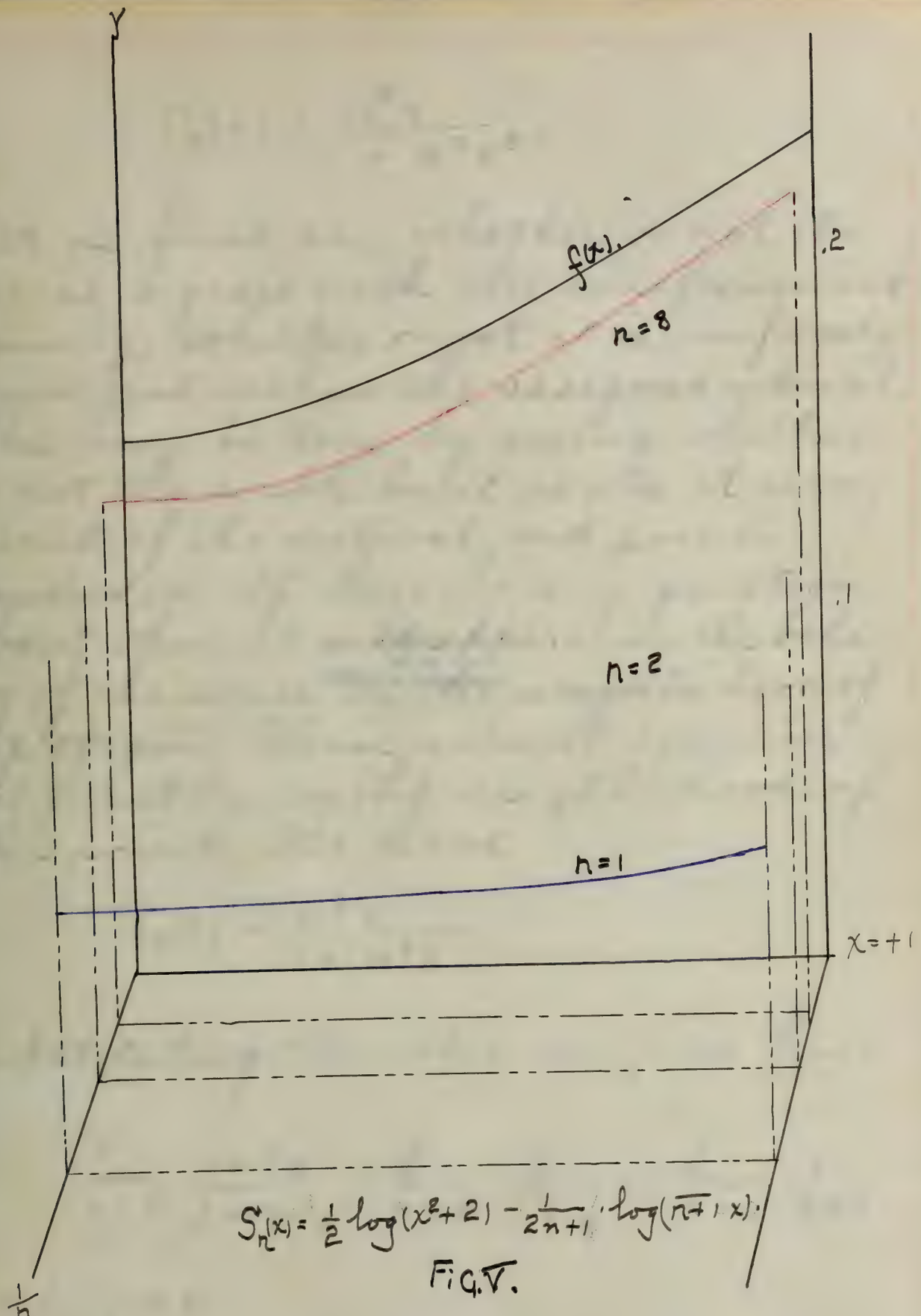
$$\int_0^1 f(x)\,dx=\lim_{n=\infty}\int_0^1 S_n(x)\,dx.$$

and again we find that the limit of the sum of the terms by terms integrals of an uniformly convergent infinite series is the same as the integral of the limit of the series. We might have anticipated this result, because our series obeyed the given condition, being continuous and uniformly convergent within a finite interval.

Example. Let us take the infinite series

$$f(x)=\frac{x}{1+x^2}+\frac{4x}{1+8x^2}-\frac{x}{1+x^2}+\ldots\ldots;\ 0\leq x\leq 1.$$

As the series is a telescoping series we see at once that




$$S_n(x) = \frac{1}{2}\log(x^2+2) - \frac{1}{2n+1}\cdot\log(n+1,x).$$

Fig. V.

$$S_n(x) = \frac{n^2 x}{1 + n^3 x^2}.$$

Let us first see whether or not the series agrees with the chief condition, namely, whether or not it is uniformly convergent within the assigned interval. This may be done by seeing whether or not the double limit exists at every point of the interval, end points included. If there be any question about this, it will appear in the behavior of the series in the neighborhood of the origin. Hence we will examine the function in the neighborhood of this point. We have

$$S_n(x) = \frac{n^2 x}{1 + n^3 x}.$$

Substituting the value $n = \frac{1}{x}$ we have

$$\lim_{\substack{x=0 \\ n=\frac{1}{x}}} \frac{n^2 x}{1 + n^3 x^2} = \lim_{x=0} \frac{\frac{1}{x}}{1 + \frac{1}{x}} = \lim_{x=0} \frac{1}{x+1}$$

$$= \frac{1}{0+1} = 1.$$

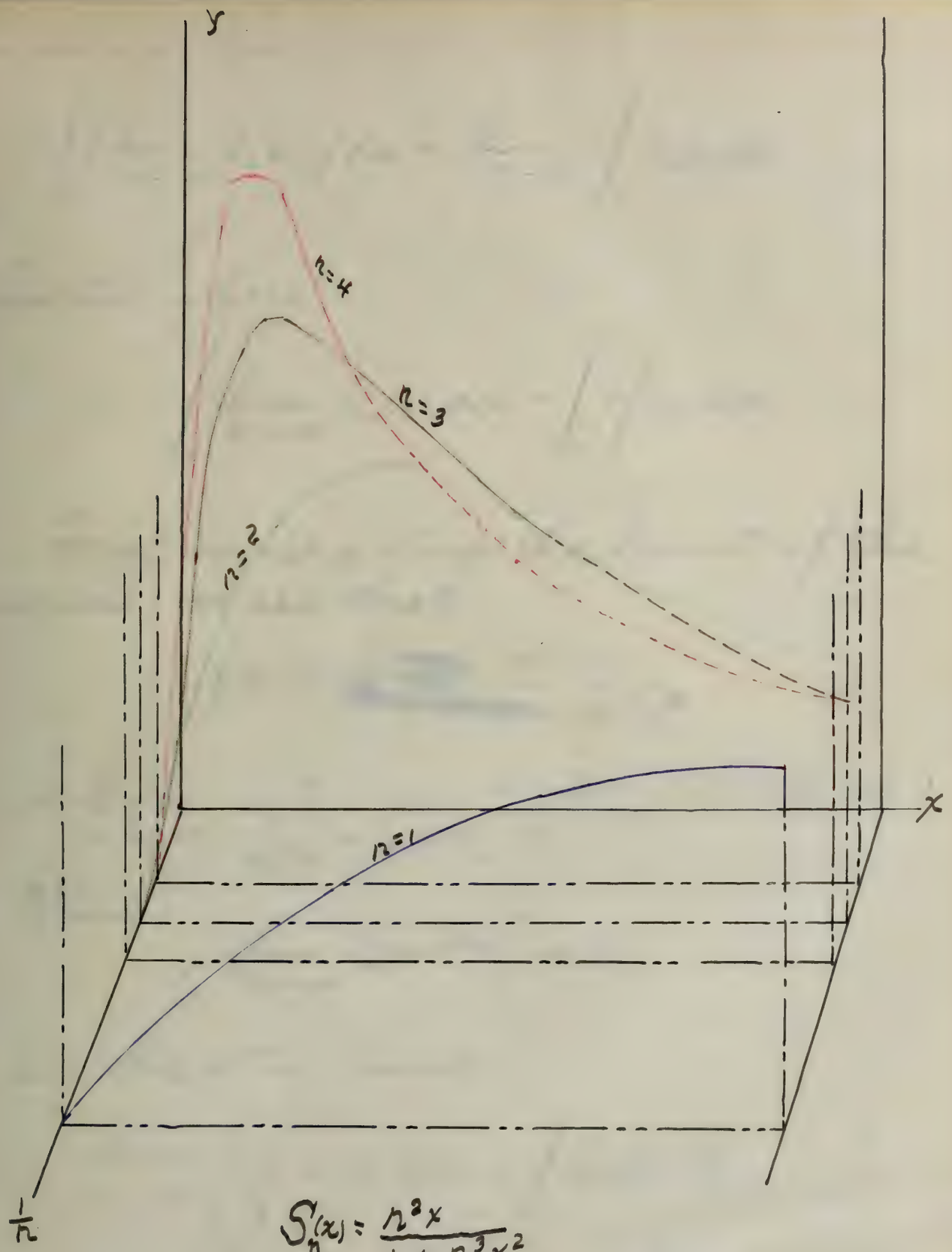
This gives one approach to origin. Again we substitute $n = x$. this gives

$$\lim_{\substack{x=0 \\ n=x}} \frac{n^2 x}{1+n^3x^2} = \lim_{x=0} \frac{1}{\frac{1}{x^3}+x^2}$$

$$= \frac{1}{\frac{1}{0}+0} = \frac{1}{\infty} = 0.$$

Hence we have two different limiting values by the two different approaches to the origin and by the theorem (b) of § 3, Chapter I, the series is non-uniformly convergent. By a geometrical consideration of the series, it is seen at once that the approximation curves of our series do not converge uniformly, for we find in each approximation curve a peak which increases in height with each succeeding curve. See Fig. VII. However, these peaks do not increase in height indefinitely but approach a finite upper limit

The question is whether or not the equation




$$S_n(x) = \frac{n^2 x}{1 + n^3 x^2}$$

FIG VII

$$\int_0^1 [\lim_{n=\infty} S_n(x)]\,dx = \lim_{n=\infty} \int_0^1 S_n(x)\,dx,$$

holds, where

$$\int_0^1 \lim_{n=\infty} S_n(x)\,dx = \int_0^1 f(x)\,dx.$$

First investigating the limit of the series, we see that

$$f(x) = \lim_{n=\infty} \frac{n^2 x}{1+n^3x^2}$$

$$= \lim_{n=\infty} \frac{x}{\frac{1}{n^2}+nx^2} = \frac{x}{0+\infty} = 0;\ 0 \leqq x \leqq 1.$$

Hence $$\int_0^1 [\lim_{n=\infty} S_n(x).dx] = 0.$$

On the other hand

$$\lim_{n=\infty} \left[\int_0^1 u_1(x)\,dx + \int_0^1 u_2(x)\,dx - \cdots\right]$$

$$= \lim_{n=\infty} \int_0^1 \frac{n^2 x.dx}{1+n^3x^2} = \lim_{n=\infty} \int_0^1 \frac{x.dx}{\frac{1}{n^2}+nx^2}$$

$$= \lim_{n \doteq \infty} \frac{1}{2n} \left[ \log (n x^2 + \frac{1}{n^2}) \right]_0^1$$

$$= \lim_{n \doteq \infty} \left[ \frac{1}{2n} \cdot \log (n + \frac{1}{n^2}) - \frac{1}{2n} \cdot \log 0 + \frac{1}{n^2} \right]$$

$$= 0$$

Thus in the case of the non-uniformly convergent series given above the limit of the sum of the integrals, term by term, is the same as the integrals of the limit of the function. This shows that the condition given above for the term by term integration of the series is a sufficient but not a necessary condition.

§ 2 A Single Peak of Finite Height. As we have seen in the previous section, the integral of the limit of an infinite convergent series may be the limit approached by the sum of the integrals taken term by term, even when the series is non-uniformly convergent, it is necessary that we study the manner in which our non-uniformly convergent series converges. We take a series defined as in § 1 of this chapter with the exception that the condition of uniform convergence is not required. It is convenient in studying the manner of convergence, to study the problem from a geometrical standpoint, for it is only by seeing the various approximation curves, which the sum of the first $n$ terms of a series represents where $n = 1, 2, 3, \cdots$, that one is able to understand the significance of the peaks and their relation to the problem in hand. It might be well to add here that where an infinite

series is defined as convergent, where the terms are continuous functions, and the series converges towards a continuous limiting function, the only irregularities which enter into the curve, corresponding to the limit of the approximation curves, are peaks. The method of studying the manner of convergence, and of studying our problem is to investigate the height and number of the peaks. This can be done most easily geometrically.

We assume our infinite series in any case to be of the form

$$f(x) = u_1(x) + u_2(x) + \cdots + u_n(x) + \cdots; a \leqq x \leqq b,$$

where the $u$'s and $f(x)$ are continuous throughout the assigned interval. The sum of the first $n$ terms is of the form

$$S_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x); n = 1, 2, 3 \cdots,$$

We have then in all cases, where $f(x)$ is non-uniformly convergent, one or more peaks in the curve representing

the limit of
the sum of the first $n$ terms of the series, that is to say in $S_n(x)$. We study the manner of change of the one or more peaks in $S_n(x)$ as $n$ increases and find that there is a certain relation between the height of the peaks and the problem with which we are concerned.

In this section we shall study single peaks of finite height. It is plain that as the peaks remain finite in height and the curve converges uniformly throughout the remaining parts of the interval toward $f(x)$, that the area under the approximation curves will approach nearer and nearer to that under the curve representing $f(x)$. The peak, for there is but one in the given interval, becomes thinner and thinner as $n$ increases and the area under the curves therefore, less and less. This leads to the following theorem.

A sufficient condition that

$$\int_a^b \lim_{n=\infty} S_n(x)\,dx = \lim_{n=\infty} \int_a^b S_n(x)\,dx,$$

where $S_n(x)$ is continuous in $x$, is that the curve representing the series have a single peak of finite height.

Example. Let us consider the infinite series

$$f(x) = \frac{x}{1+x^2} + \left(\frac{2x}{1+4x^2} - \frac{x}{1+x^2}\right) + \cdots; \quad 0 \leqq x \leqq 1.$$

From this we see that the series is a telescopic series, and that

$$S_n(x) = \frac{nx}{1+n^2x^2}; \quad n = 1, 2, 3, \cdots.$$

Testing for continuity, we see that $S_n(x)$ is continuous in $(x)$ for all finite values of $n$ which is equivalent to saying that the $u(x)$'s are continuous in $x$. That this series is non-uniformly convergent can be easily proved by the methods given in (b) § 3, Chapter I, of this thesis. However the nonuniform convergence is also seen from the approximation curves as shown in the following Fig., and besides it is understood that the condition for the existence

of a peak is the non-uniform convergence of the series. It is seen that peaks in this series are finite and always of the same height, that is $\frac{1}{2}$. Furthermore there is but one and that occurs in the neighborhood of the origin. We see that the function represented by the series

$$f(x) = \lim_{n=\infty} S_n(x) = \lim_{n=\infty} \left( \frac{nx}{1+n^2x^2} \right)$$

$$= \lim_{n=\infty} \left( \frac{1}{\frac{1}{nx} + nx} \right) = \frac{1}{0+\infty} = 0.$$

The function is therefore continuous in the given interval, both of the conditions as to continuity being fulfilled.

To find the limit approached by the sum of the term by term integrals of the series, within the assigned interval, as $n$ approaches infinity, we proceed as follows. We have

$$\int_0' S_n(x)\,dx = \int_0' \frac{nx\,dx}{1+n^2x^2} = \int_0' \frac{x\,dx}{\frac{1}{n} + nx^2}$$

$$= \left[ \frac{1}{2n} \cdot \log\left(x^2 + \frac{1}{n^2}\right) \right]_0' = \frac{1}{2n} \cdot \log\left(1 + \frac{1}{n^2}\right)$$

$$- \frac{1}{2n} \cdot \log\left(0 + \frac{1}{n^2}\right) =$$

Letting $n$ increase indefinitely, we find that

$$\lim_{n=\infty} \frac{1}{2n} \left( \log\left(1 + \frac{1}{n^2}\right) - \log\left(0 + \frac{1}{n^2}\right) \right) = 0$$

Hence we find that

$$\int_0' f(x)\,dx = \lim_{n=\infty} \int_0' S_n(x)\,dx = 0.$$

This is also seen geometrically, for while the peaks remain constant in height the other parts of the curve approach uniformly the x axis as n increases. Thus we see that the area under the curve must decrease toward the limit zero as n approaches infinity.

Example. Let us consider the infinite series

$$f(x) = \frac{2x}{1+x} + \left(\frac{4x}{1+4x} - \frac{2x}{1+x}\right) + \left(\frac{15x}{3+27x} - \frac{4x}{1+4x}\right) + \cdots$$

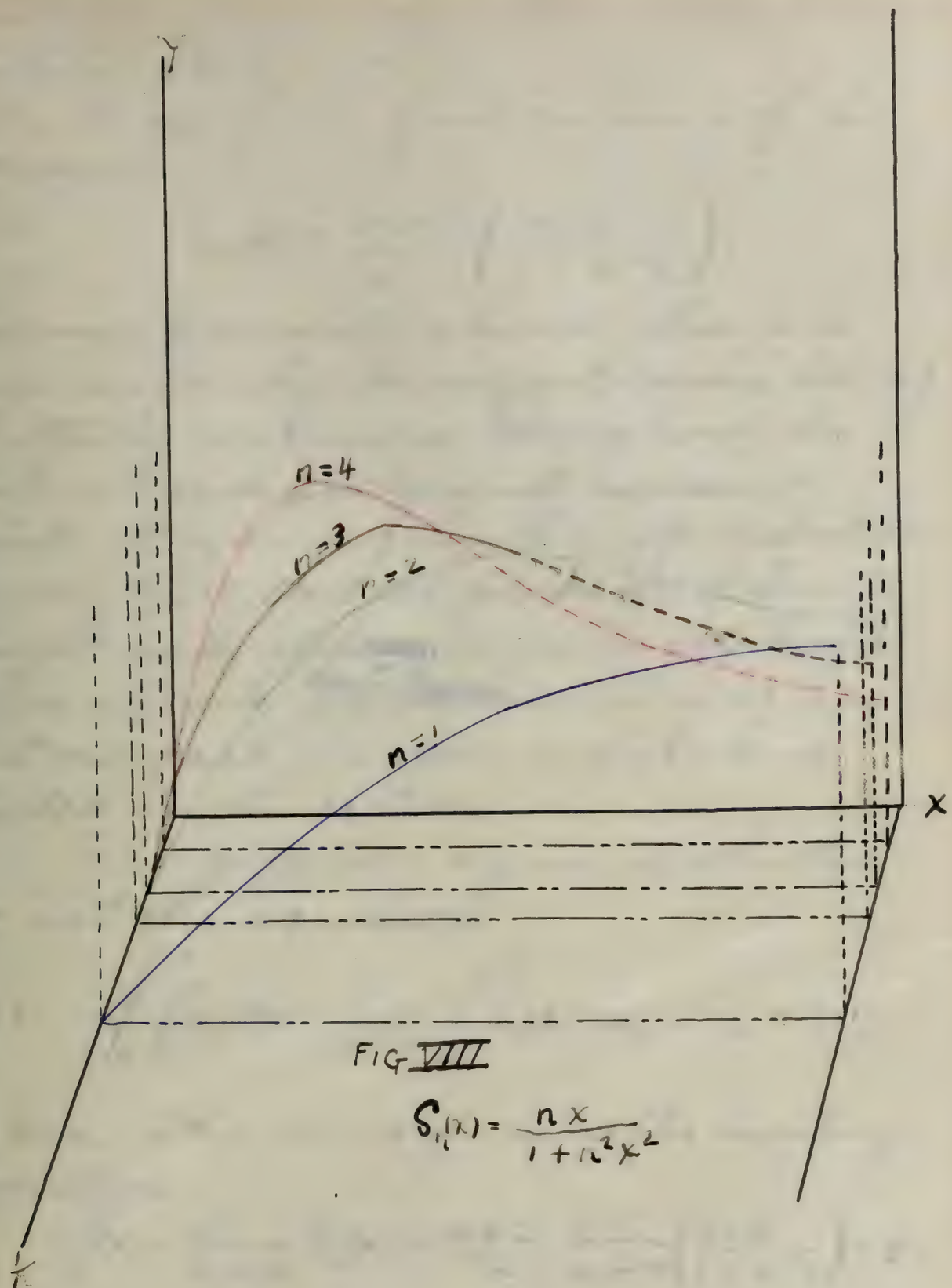



FIG VIII

$$S_{1_k}(x) = \frac{nx}{1+k^2x^2}$$

where $0 \leqq x \leqq 1$.

The sum of the first $n$ terms of the series is

$$S_n(x) = \frac{n+2}{n}\left(\frac{nx}{1+n^2x}\right).$$

As was previously shown, this is a non-uniformly convergent series, having the terms continuous throughout the interval and peaks must necessarily exist. It is seen from the approximation curves, FIG IV, that the peaks are of finite height and approach a lower limit* Thus as each approximation curve has but one peak the series is of the kind treated in this section

The question again is whether or not the equation

$$(1) \qquad \int_0^1 f(x)\,dx = \lim_{n=\infty} \int_0^1 S_n(x)\,dx; \quad 0 \leqq x \leqq 1,$$

is true. The series gives us the limiting function

$$f(x) = \lim_{n=\infty} S_n(x) = \frac{n+2}{n} \cdot \lim_{n=\infty}\left(\frac{nx}{1+n^2x}\right) = 0.$$

---

* See page 16, FIG IV, Chapter I, of this Thesis.

On the other hand we have

$$\int_0^1 S_n(x)\,dx = \int_0^1 \frac{n+2}{n}\left(\frac{n.x\,dx}{1+n^2x}\right)$$

$$= \frac{n+2}{n}\int_0^1 \frac{x\,dx}{\frac{1}{n}+nx}$$

$$= \frac{1}{n^2}\left[\log\left(\frac{1}{n}+n\right) + \frac{\frac{1}{n}}{\frac{1}{n}+n} - \log\left(\frac{1}{n}+n.0\right) + \frac{\frac{1}{n}}{\frac{1}{n}+n.0}\right].$$

But as $n$ increases we have

$$\lim_{n=\infty} \frac{1}{n^2}\left[\log\left(\frac{1}{n}+n\right) + \frac{\frac{1}{n}}{\frac{1}{n}+n} - \log\left(\frac{1}{n}\right) + 1\right) = 0,^*$$

and therefore,

$$\int_0^1 f(x)\,dx = \lim_{n=\infty} \int_0^1 S_n(x)\,dx = 0.\ ;\ 0 \leqq x \leqq 1.$$

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* See example 13, page 26 of Pascal's "Esercizi Di Calcolo Infinitesimale"

§ 3. 17. Finite Number of Finite Peaks. The assumption here is that the number of peaks is not only finite but remains less than some definite finite number as $n$ increases indefinitely. As we have seen in the previous section, that the integral of the limit of an infinite, convergent series is equal to the sum of the term by term integrals, when the terms of the series are continuous and the curve representing the limit of the approximation curves has but a single finite peak. It follows as a matter of course, that we should study the series which has a finite number of finite peaks. Again it is convenient that the problem be studied geometrically. The relation which a finite number of finite peaks bears to our problem in hand is most easily shown by means of the approximation curves which represent the sum of the first $n$ terms of the series, that is, where $n = 1, 2, 3, \ldots\ldots$.

Our series is as previously defined,

$$f(x) = u_1(x) + u_2(x) + \cdots + u_n(x) + \cdots ; a \leqq x \leqq b,$$

and as before the sum of the first $n$ terms is

$$S_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x).$$

We have in each approximation curve, from some definite point on, a finite number of finite peaks, and the method of dealing with the question is to study the changes in form and in area of these peaks as $n$ increases indefinitely. We are able to follow very closely the method of the previous section, where we dealt with a single finite peak. The curve converges uniformly throughout the remaining parts of the interval, toward the the limiting function, $f(x)$, and the area under the approximation curves will approach nearer and nearer to that under the curve representing $f(x)$. The peaks, remaining always less than some finite number $G$, become at the

same time thinner and thinner as $n$ increases, consequently the area under the approximation curves differs therefore less and less as $n$ increases from that under the curve representing $f(x)$. The following theorem results directly.

A sufficient condition that

$$\int_a^b \lim_{n=\infty} S_n(x)\,dx = \lim_{n=\infty} \int_a^b S_n(x)\,dx$$

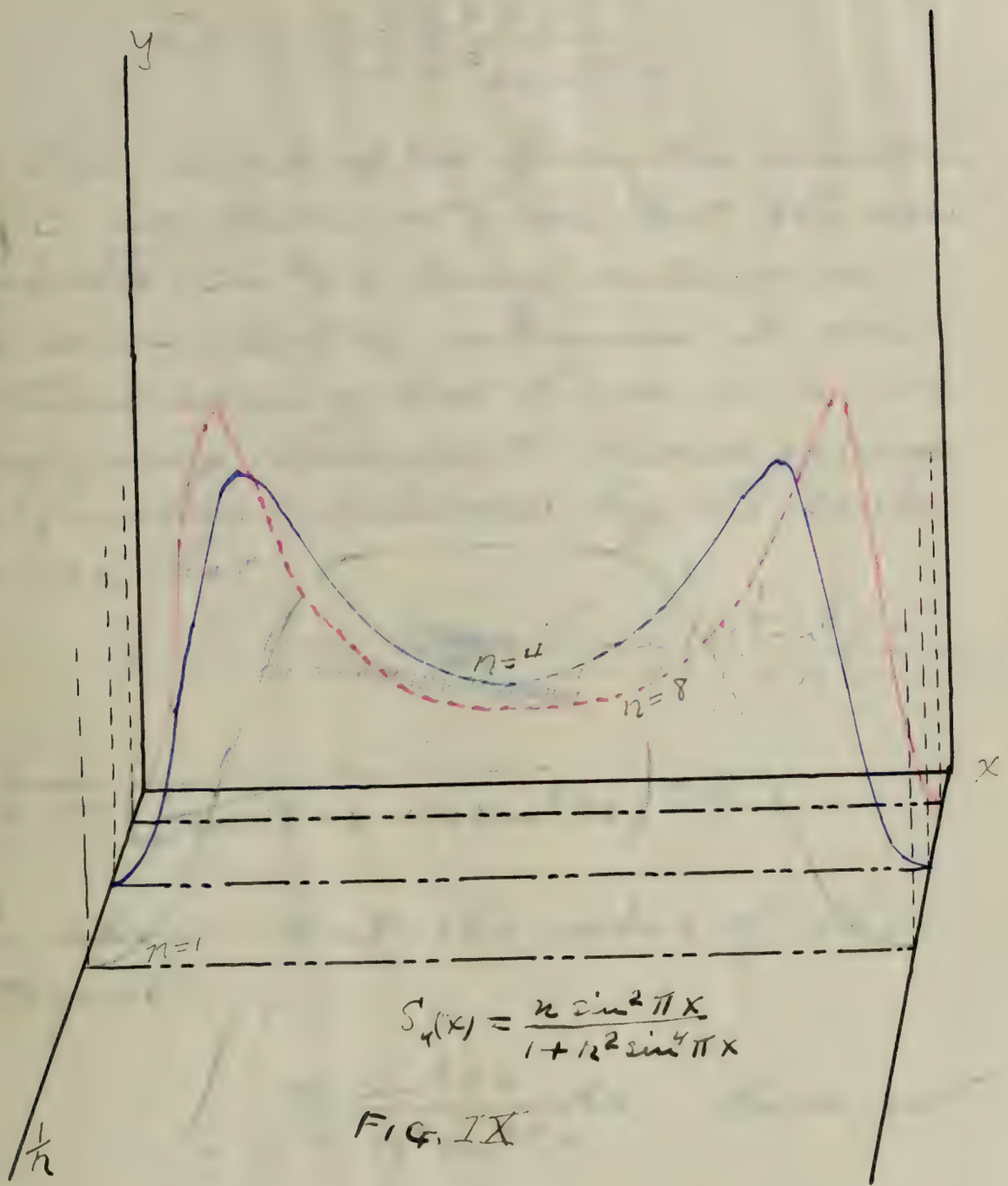
where $S_n(x)$ is continuous in $(x)$, is that the curve representing the limit of the approximation curves has a finite number of finite peaks.

Example. Let us consider the infinite series

$$f(x) = \frac{\sin^2 \pi x}{1+\sin^4 \pi x} + \left(\frac{2\sin^2 \pi x}{1+4\sin^4 \pi x} - \frac{\sin^2 \pi x}{1+\sin^4 \pi x}\right)$$

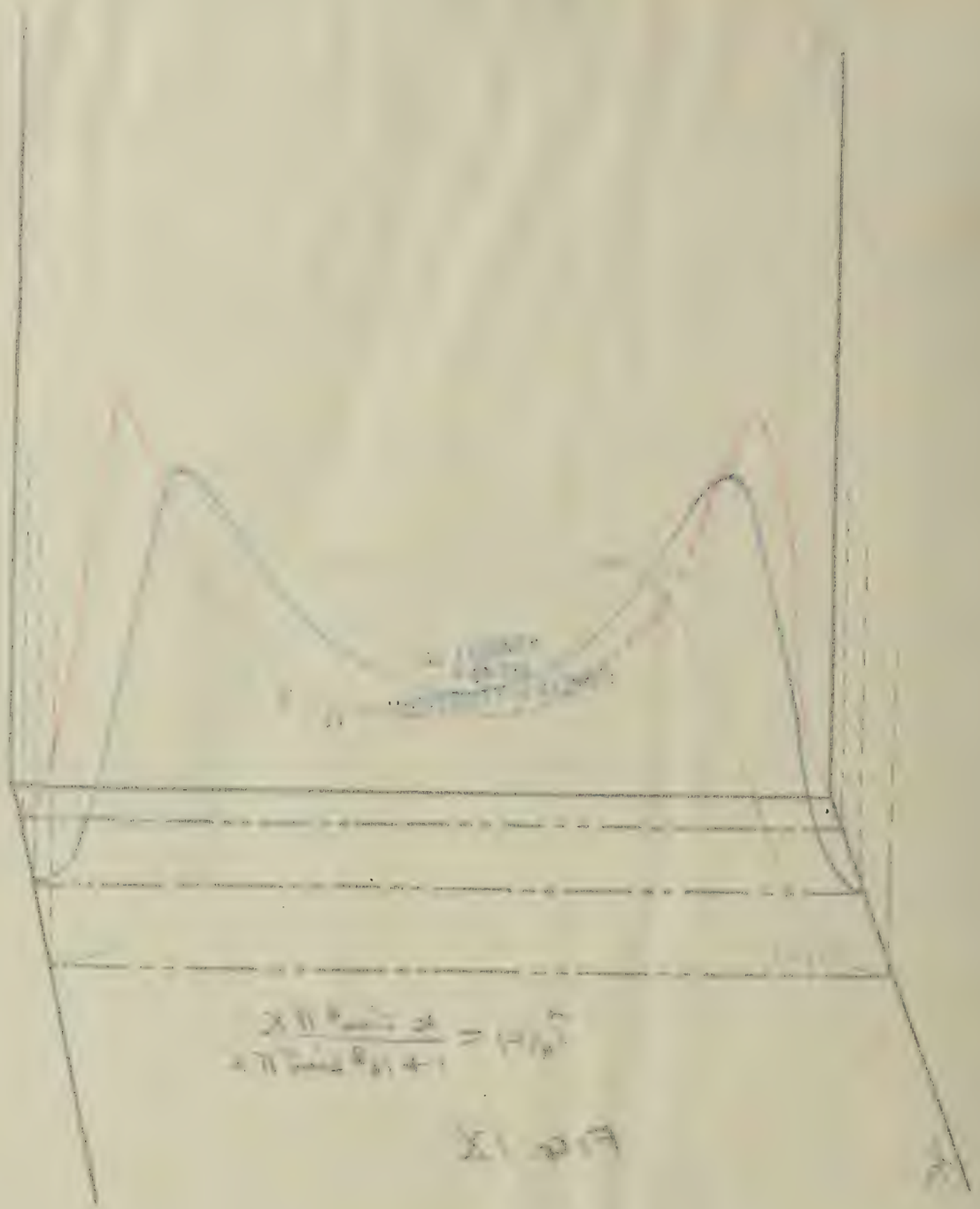
$$+ \cdots + \left(\frac{n\sin^2 \pi x}{1+n^2\sin^4 \pi x} - \frac{(n-1)\sin^2 \pi x}{1+(n-1)^2\sin^4 \pi x}\right) + \cdots ; \quad 0 \leqq x \leqq 1,$$

In this series we have




$$S_n(x) = \frac{n \sin^2 \pi x}{1 + n^2 \sin^4 \pi x}$$

FIG. IX



$$S_n(x) = \frac{n \sin^2 \pi x}{1 + n^2 \sin^4 \pi x}.$$

As this is adapted from the example 1, § 2, we know at once that, the new variable $\sin^2 \pi x$ being continuous, the series itself is continuous. It goes without saying that the series is non-uniformly convergent. Investigating the function represented by the series, we see that

$$f(x) = \lim_{n=\infty} S_n(x) = \lim_{n=\infty} \left( \frac{n \sin^2 \pi x}{1 + n^2 \sin^4 \pi x} \right)$$

$$= \lim_{n=\infty} \left( \frac{1}{\frac{1}{n} \sin^2 2\pi x + n \sin^2 \pi x} \right) = 0.$$

To show that the value of the integral

$$\int_0^1 \frac{n \sin^2 \pi x}{1 + n^2 \sin^4 \pi x} . dx$$ does not

differ from

$$\int_0^1 f(x)\, dx = 0,$$

we proceed geometrically and as follows.

Since the function to be integrated is a continuous function, we know from the properties of such functions that the given definite integral must exist. We will denote it by $F(x)$. We will include each peak within a sub-interval of length not greater than the arbitrarily small positive number $\epsilon$. This is always possible by taking $n$ sufficiently large. We have then

$$\int_0^1 \frac{n \sin^2 \pi x \, dx}{1 + n^2 \sin^4 \pi x} = F(x) \leqq \int_0^1 f(x) dx + G \cdot \frac{\epsilon}{2}$$

when $G$ is equal to or greater than the number of peaks and $\frac{1}{2}$ is the height of the peaks. However, we can make $\epsilon$ as small as we chose by making $n$ sufficiently large. Hence in the limit as $n \doteq \infty$, we have

$$\lim_{n \doteq \infty} \int_0^1 \frac{n \sin^2 \pi x}{1 + n^2 \sin^4 \pi x} \cdot dx = \lim_{n \doteq \infty} F(x) \leqq \int_0^1 f(x) \cdot dx = 0,$$

but as the integral, or area, must be positive, we may write

$$\lim_{n \doteq \infty} F(x) = \int_0^1 f(x) dx,$$

which is the required result, and shows that the given series can be integrated term by term. This is also seen geometrically in Fig IX. The peaks remain constant in height while the other parts of the curve approach uniformly the x axis as n increases. Thus we see that the area under the approximation curves must decrease towards the limit zero.

§ 4. Infinite Number of Finite Peaks. We now proceed to consider the case which is the natural sequence of § 3, of this chapter. We have found that it is possible to integrate an infinite series term by term, when that series is convergent within a finite interval and is represented by a continuous curve which has a finite number of finite peaks. We next consider the problem of the thesis as applied to a series which is represented by a continuous curve which has an infinite number of finite peaks. There are several difficulties which make themselves apparent when one begins to discuss a series of this sort. First, a telescopic series cannot be used, because we can not have a telescopic series of the kind here defined, which will converge within a finite interval and still be a function of two variables. Second, where we have a series of an infinite number of terms, the same in sign, it is necessary that the heights of the

peaks as $n$ increases must approach a lower limit in such a way as to render our entire series convergent and represent a continuous function. See FIGS. X and XI. So in case we can construct such a series, that is an infinite, continuous, convergent series, which is represented by a curve which has an infinite number of finite peaks, the geometrical proof should follow easily.

Let our series be of the form

$$f(x) = u_1(x) + u_2(x) + \cdots + u_n(x) + \cdots; \quad a \leqq x \leqq b,$$

and

$$S_n(x) = u_1(x) + u_2(x) + u_3(x) + \cdots u_n(x),$$

where $u_n(x)$ and hence $S_n(x)$, is a continuous function of $x$, and where the same is also true of $f(x)$. Let us suppose that in the limit the approximation curves have an infinite number of peaks which are finite. These peaks may in fact occur in the neighborhood of a set of points everywhere dense in the given inter-

val, but points must also exist and be every dense in the same interval in the neighborhood of which no peaks occur which are greater in height than an arbitrarily small $\sigma$.* Since all of the peaks are of finite height, say $< B$, we may then represent the condition of affairs by the figure, see FIG X, where the curve is

$$y = R_n(x) \equiv (f(x) - S_n(x)),$$

where we are to consider the area of that portion of the peaks above $y = \sigma$. We may write

$$(1) \qquad -\varepsilon < R_n(x) < \varepsilon$$

for every value of $n$ greater than some value, say $n = m$. For, if this were not the case one of these inequalities in (1) would be violated, say the one on the right, for an infinite number of values of $n$, say $n = n_1, n_2, n_3, \ldots\ldots$ Choose

$$\sigma < \frac{\varepsilon}{b-a}.$$

Then those portions of the approximation

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* See Osgood American Journal of Mathematics Vol. 19, 176

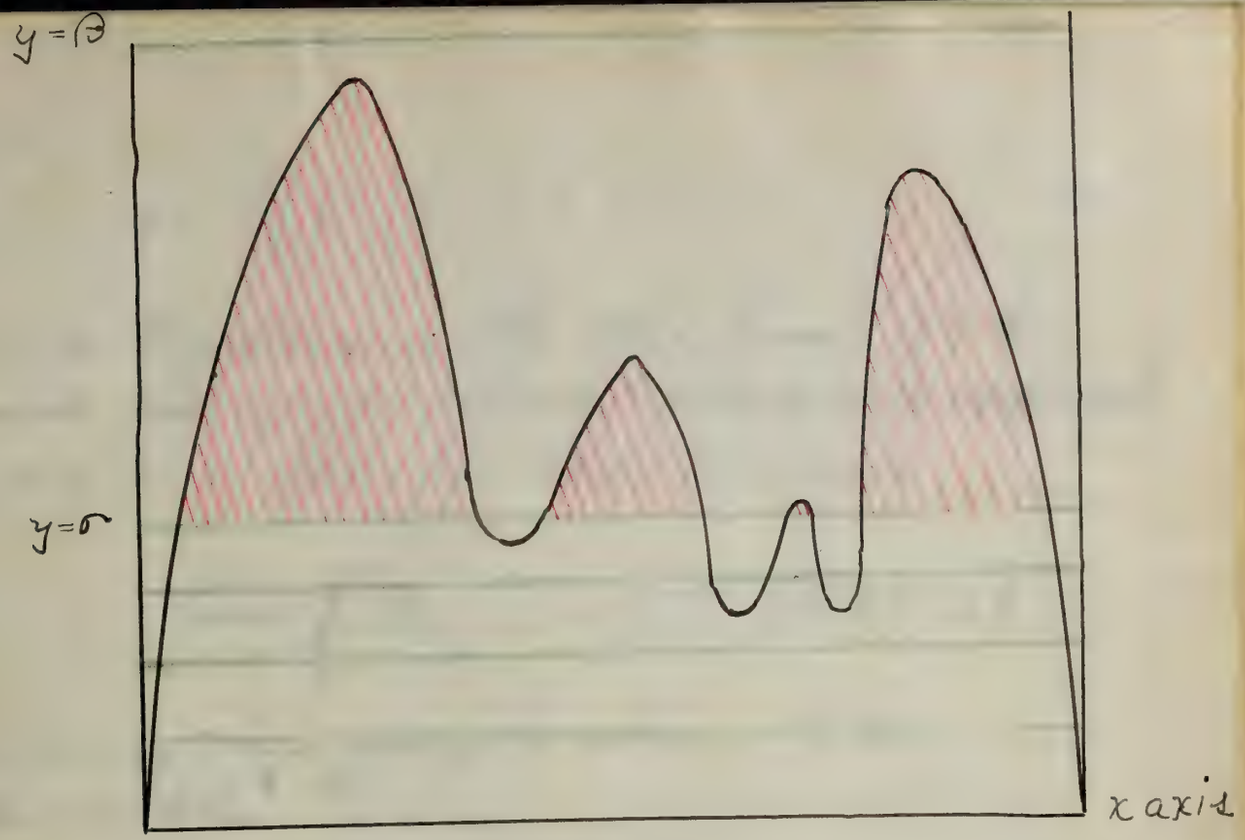
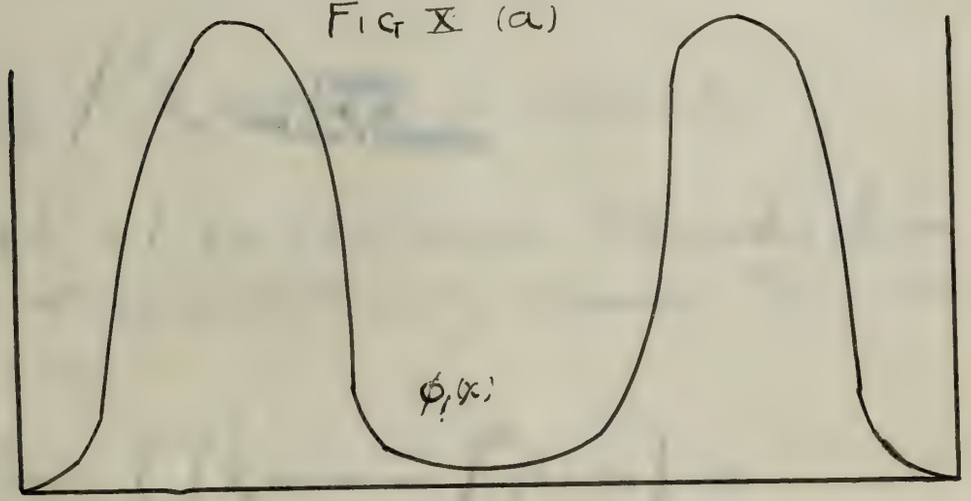
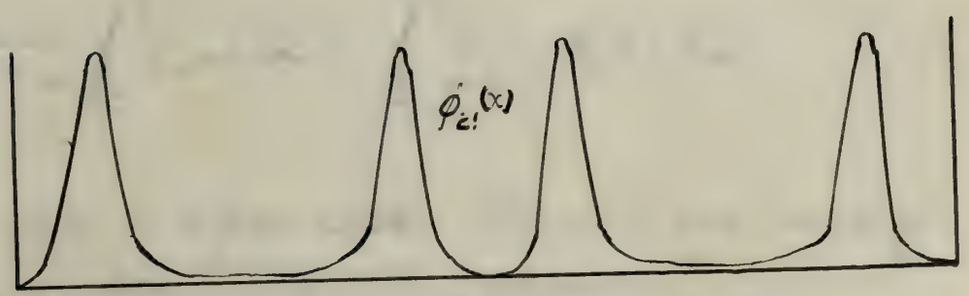
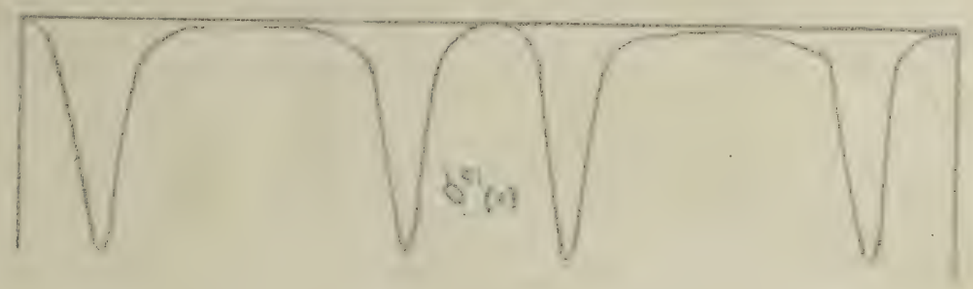


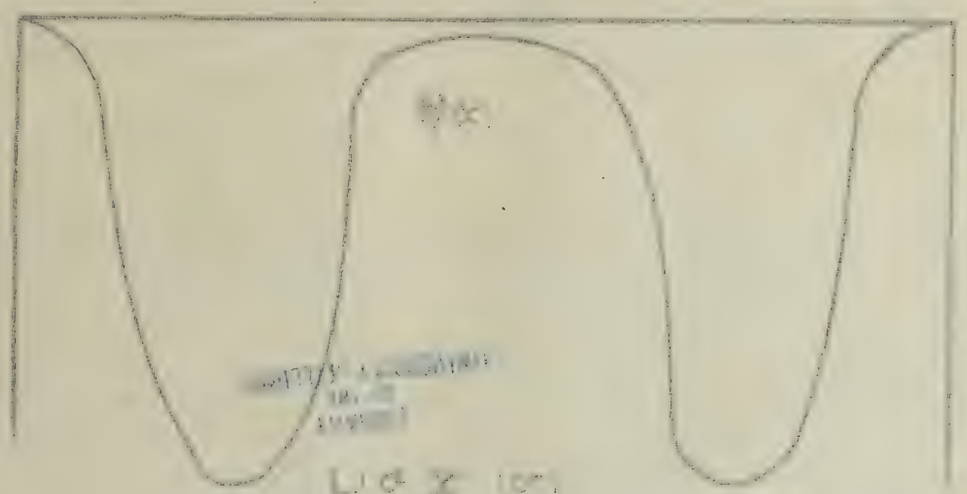

FIG X (a)

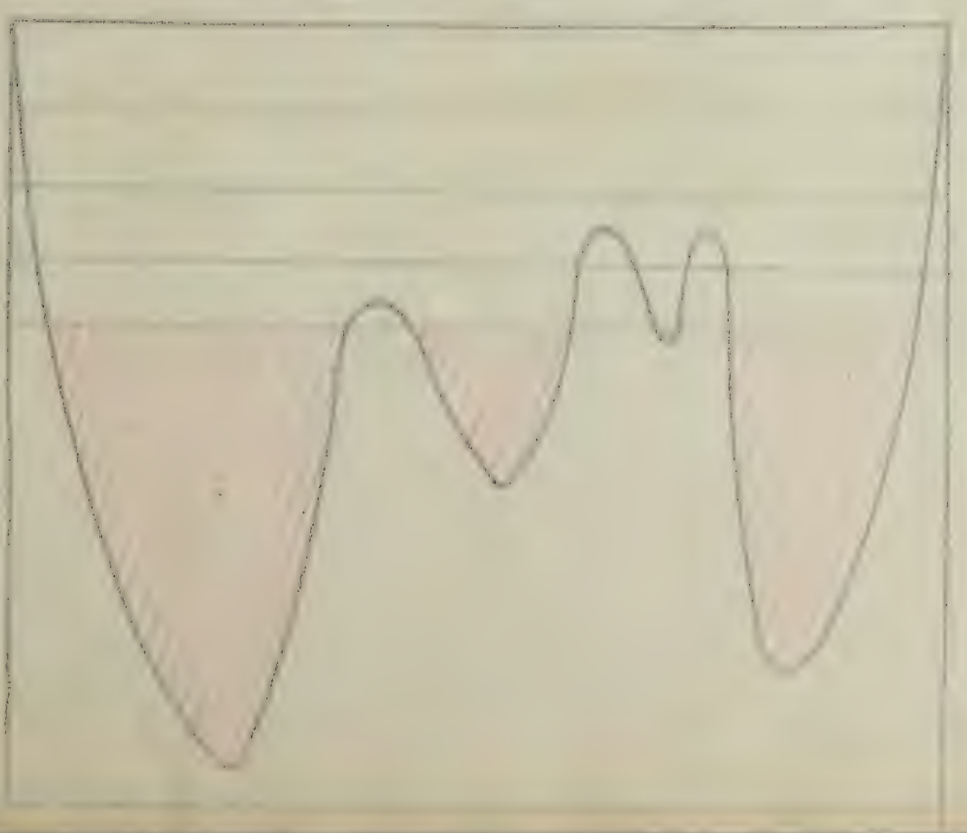



FIGS. X (b)







curves

$$y = R_{n_i}(x) \; ; \; i = 1, 2, 3, \ldots$$

would together with the line $y = \sigma$ bound an area which could not approach 0 as $i \doteq \infty$; for, we should have

$$\lim_{i \doteq \infty} \int_a^b R_{n_i}(x).dx \geqq \varepsilon > \sigma(b-a).$$

However, this may be shown to be impossible.* Hence

$$\lim_{i \doteq \infty} \int_a^b R_{n_i}(x)dx < \varepsilon > \sigma(b-a),$$

when $\sigma(b-a)$ is the area below the line $y = \sigma$ and $\sigma$ is arbitrarily small. Hence we may write

$$\lim_{n \doteq \infty} \left\{ \int_a^b S_n(x).dx - \int_a^b f(x).dx \right\} = 0$$

or

$$\lim_{n \doteq \infty} \int_a^b S_n(x)\,dx = \int_a^b \lim_{n \doteq \infty} S_n(x)\,dx$$

which was required. Hence we may state the following theorem:

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*See Osgood American Journal of Mathematics, Vol. 19, p. 180.

If $S_n(x)$ is continuous and converges toward a continuous function $f(x)$ in the interval $(a,b)$, The approximation curves having in the limit only finite peaks, although these may be infinite in number, then

$$\int_a^b \lim_{n=\infty} S_n(x)\,dx = \lim_{n=\infty} \int_a^b S_n(x)\,dx;$$

that is, the series

$$f(x) = u_1(x) + u_2(x) + u_3(x) + \cdots + u_n(x) + \cdots$$

may be integrated term by term.

Example. As an example where this criterion will prove of service, let us construct the following series by means of Hankel's principle of condensation. Commence with the function

$$\frac{nx}{1 + n^2x^2}$$

which we have already studied and have seen that it has a single peak of height $\frac{1}{2}$ in the neighborhood of $x = 0$. Putting $\sin^2 \pi x$ in the place of $x$ we get

have

$$\phi_\kappa = \frac{u \sin^2 \pi \kappa x}{1 + u^2 \sin^4 \pi \kappa x}; \quad \kappa = 1, 2, 3, \cdots,$$

which will have peaks in the neighborhood of $0, \frac{1}{\kappa}, \frac{2}{\kappa}, \frac{3}{\kappa} \cdots; \frac{\kappa}{\kappa} = 1.$

If we now consider $S_u(x)$ as the series

$$S_u(x) = \phi_1(x) + \frac{1}{2!}\phi_2 x + \cdots \equiv \sum_{\kappa=1}^{\infty} \frac{1}{\kappa!} \phi_\kappa(x),$$

we have in the limit of our approximation curve peaks in the neighborhood of an infinite number of points, namely in the neighborhood of all rational numbers. Since the series, as a function in $u, x$, is uniformly convergent, we may write

$$f(x) = \lim_{u \doteq \infty} S_u(x) = \lim_{u \doteq \infty} \phi_1(x) + \lim_{u \doteq \infty} \frac{\phi_2(x)}{2!} + \cdots.$$

The peaks contributed to the approximation curves $S_u(x)$ by the addition of any new term, say the $\kappa^{\text{th}}$ term, cannot be larger than $\frac{1}{\kappa!}$ since $\phi_{\kappa!}(x)$ is always less than one. Hence some of the peaks are <u>very</u> small as shown by the accompanying

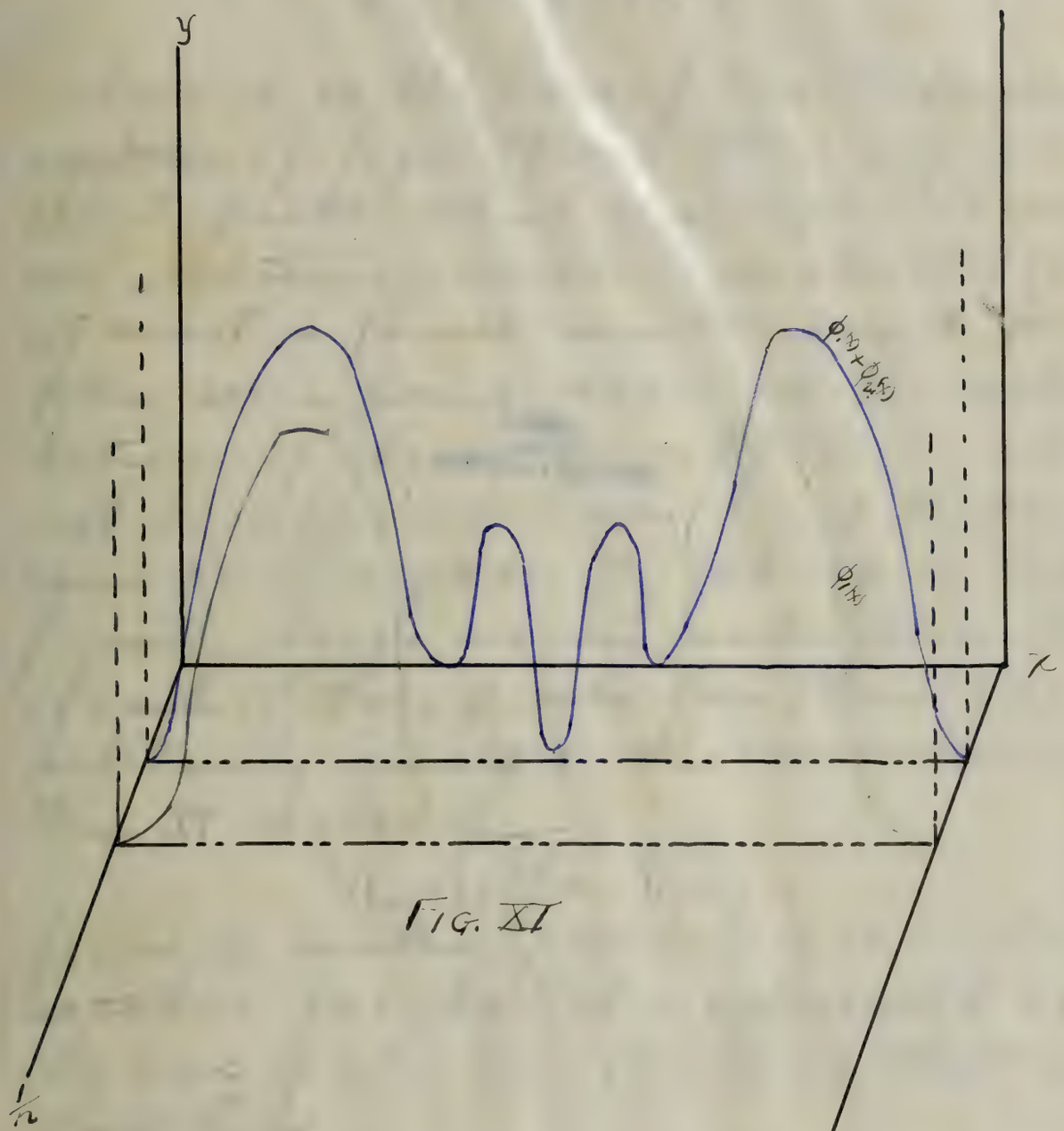


FIG. XI

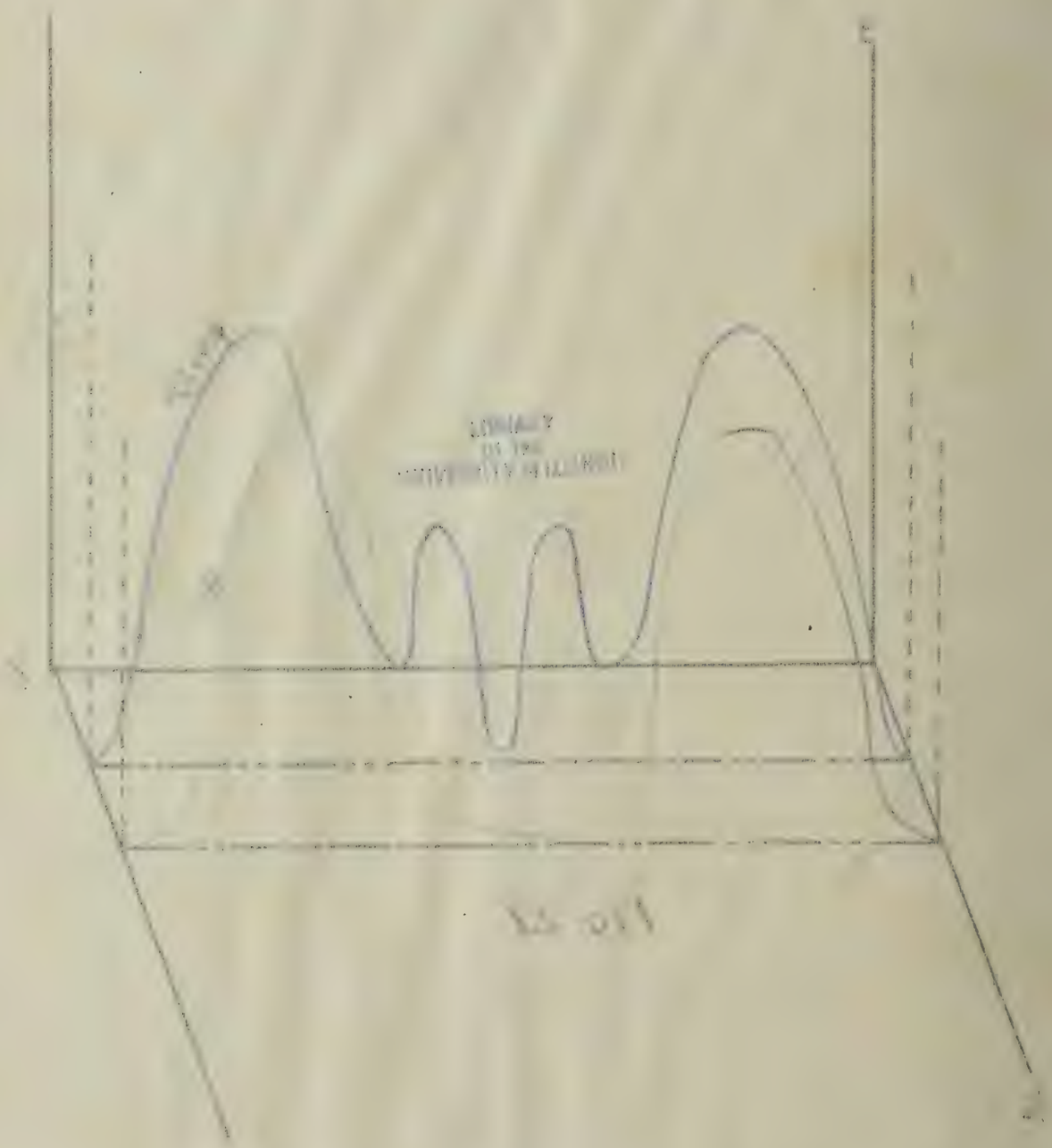

figure XI.

Moreover, the height of any peak cannot be greater than

$$\leq \frac{1}{n_i} = e - 1 < 2$$

where $e$ is the base of the Naperian system of logarithms. Hence if we select peaks which are higher than an arbitrarily small $\sigma$, we shall have at most a finite number say $N$, because from some point $n = m$ on the contributions of the new terms of the series will not be sufficient to even bring the peak above the line $y = \sigma$. By taking $n$ sufficiently large we can make the width of each of these peaks less than an arbitrarily small $\epsilon$. The curve shown in Fig. XI is that of

$$R_n(x) = \left\{ f(x) - S_n(x) \right\},$$

for, as a matter of convenience, we have so taken $f(x)$ that it is represented by the line $y = 0$. In order then that we shall have

$$\int_0^1 f(x)\,dx = \lim_{n=\infty} \int_0^1 S_n(x)\,dx,$$

we show that

$$\int_0^1 R_n(x)\,dx = 0,$$

but this last integral represents the area under the curve $R_n(x)$ which is made up of two parts, namely, that under the line $y = \sigma$ and that of the finite number of peaks of width $\varepsilon$ above $y = \sigma$. We may then write

$$\int_0^1 R_n(x)\,dx < \sigma(1-0) + \varepsilon N.$$

Therefore

$$\lim_{n=\infty} \int_0^1 R_n(x)\,dx < \sigma,$$

where $\sigma$ is an arbitrarily small quantity. Hence we must have

$$\lim_{n=\infty} \int_0^1 R_n(x)\,dx = 0,$$

since it can not be negative. From this our required result follows, namely, that

$$\int_0^1 \lim_{n=\infty} S_n(x)\,dx = \lim_{n=\infty} \int_0^1 S_n(x)\,dx.$$

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